

# Constraint Satisfaction Problems

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## Example: Job-shop scheduling

- Consider a small part of the car assembly, consists of 15 tasks
  - Install axles (front & back)
  - Affix all four wheels (right and left, front and back)
  - Tighten nuts for each wheel, affix hubcaps
  - Inspect the final assembly
- Represent the tasks with 15 variables:

$$X = \{Axle_F, Axle_B, Wheel_{RF}, Wheel_{LF}, Wheel_{RB}, Wheel_{LB}, Nuts_{RF}, Nuts_{LF}, Nuts_{RB}, Nuts_{LB}, Cap_{RF}, Cap_{LF}, Cap_{RB}, Cap_{LB}, Inspect\}$$

- The value of each variable: the time that tasks starts
- **Precedence constraints** (between individual tasks)

$$T_1 + d_1 \leq T_2$$

$T_1$ must occur before $T_2$ . $T_1$ takes duration $d_1$ to complete.
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## Example: Job-shop scheduling

- Install the axles (10 min): They have to be in place before the wheels are put on

$$\begin{array}{ll} Axle_F + 10 \leq Wheel_{RF} & Axle_F + 10 \leq Wheel_{LF} \\ Axle_B + 10 \leq Wheel_{RB} & Axle_B + 10 \leq Wheel_{LB} \end{array}$$

- Affix the wheels (1 min), tighten the nuts (2 min), and attach the hubcap (1 min):

$$\begin{array}{ll} Wheel_{RF} + 1 \leq Nuts_{RF} & Nuts_{RF} + 2 \leq Cap_{RF} \\ Wheel_{LF} + 1 \leq Nuts_{LF} & Nuts_{LF} + 2 \leq Cap_{LF} \\ Wheel_{RB} + 1 \leq Nuts_{RB} & Nuts_{RB} + 2 \leq Cap_{RB} \\ Wheel_{LB} + 1 \leq Nuts_{LB} & Nuts_{LB} + 2 \leq Cap_{LB} \end{array}$$

## Example: Job-shop scheduling

- **Disjunctive constraint:**  $Axle_F$  and  $Axle_B$  must not overlap in time
  - Either one comes first and the other does

$$(Axle_F + 10 \leq Axle_B) \text{ or } (Axle_B + 10 \leq Axle_F)$$

- Do inspection (3 min for each task): for every variable except *Inspect*, we add a constraint:

$$X + d_X \leq \text{Inspect}$$

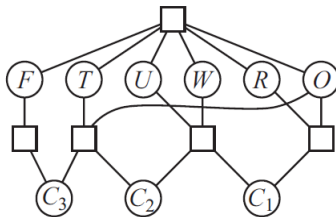
- Limit the domain of all variables (The whole assembly should be done in 30 min):

$$D_i = \{1, 2, 3, \dots, 27\} \tag{1}$$

# Constraints

- **Unary constraint:** restricts the value of a single variable
- **Binary constraint:** relates two variables
- **Global constraint:** involving an arbitrary number of variables
  - E.g.) **Cryptarithmic** puzzles -  $Alldiff(F, T, U, R, O)$

$$\begin{array}{r} T W O \\ + T W O \\ \hline F O U R \end{array}$$



$$O + O = R + 10 \cdot C_{10}$$

$$C_{10} + W + W = U + 10 \cdot C_{100}$$

$$C_{100} + T + T = O + 10 \cdot C_{1000}$$

$$C_{1000} = F$$

Additional constraints can be represented by **hypergraph**.

# Constraint propagation: Inference in CSPs

- CSP algorithm has a choice between:
- **Search:** Choose a new variable assignment
- **Constraint propagation:** using the constraints to reduce the numbers of legal values for variables
  - may be intertwined with search, or may be done as a preprocessing step, before search starts.

# Constraint propagation: Local consistency

- **Local consistency:** Enforcing local consistency in each node and arc of a constraint graph eliminate inconsistent values throughout the graph.
- **Node consistency:** Satisfy the variables's unary constraints on their values. E.g.)  $\langle (SA), SA \neq \text{green} \rangle$
- **Arc consistency:** Satisfy the variables's binary constraints.
  - For the constraint  $Y = X^2$ , we have  $\langle (X, Y), \{(0, 0), (1, 1), (2, 4), (3, 9)\} \rangle$
  - But, For the map-coloring problem, arc consistency “do nothing”:

$$\langle (SA, WA), \{(R, G), (R, B), (G, R), (G, B), (B, R), (B, G)\} \rangle$$

No matter what value you choose for  $SA$ , there is a valid value for  $WA$ .

## Arc-consistent algorithm – AC-3

- After applying AC-3, either every arc is arc-consistent, or some variable has an empty domain, indicating that the CSP cannot be solved.

**function** AC-3(*csp*) **returns** false if an inconsistency is found and true otherwise

**inputs:** *csp*, a binary CSP with components ( $X$ ,  $D$ ,  $C$ )

**local variables:** *queue*, a queue of arcs, initially all the arcs in *csp*

**while** *queue* is not empty **do**

$(X_i, X_j) \leftarrow \text{REMOVE-FIRST}(\textit{queue})$

**if** REVISE(*csp*,  $X_i$ ,  $X_j$ ) **then**

**if** size of  $D_i = 0$  **then return** false

**for each**  $X_k$  **in**  $X_i.\text{NEIGHBORS} - \{X_j\}$  **do**

            add  $(X_k, X_i)$  to *queue*

**return** true

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**function** REVISE(*csp*,  $X_i$ ,  $X_j$ ) **returns** true iff we revise the domain of  $X_i$

*revised*  $\leftarrow$  false

**for each**  $x$  **in**  $D_i$  **do**

**if** no value  $y$  in  $D_j$  allows  $(x, y)$  to satisfy the constraint between  $X_i$  and  $X_j$  **then**

            delete  $x$  from  $D_i$

*revised*  $\leftarrow$  true

**return** *revised*



# Path consistency

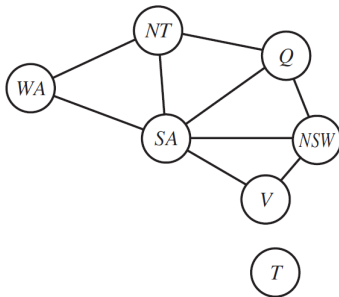
- Arc consistency: tightens down the domains (unary constraints) using the arcs (binary constraints)
- **Path consistency**: tightens the binary constraints by using implicit constraints that are inferred by looking at triples of variables.

## Path consistency

A two-variable set  $\{X_i, X_j\}$  is **path-consistent** with respect to a third variable  $X_m$  if, for every assignment  $\{X_i = a, X_j = b\}$  consistent with the constraints on  $\{X_i, X_j\}$ , there is an assignment to  $X_m$  that satisfies the constraints on  $\{X_i, X_m\}$  and  $\{X_m, X_j\}$ .

# Path consistency: Example

- Coloring the Australia map with two colors.
- What is the set  $\{WA, SA\}$  path consistent with respect to  $NT$ ?



- There are only two:  $\{WA = red, SA = blue\}$  and  $\{WA = blue, SA = red\}$
- No valid choice for  $NT$ . No solution to this problem

## K-consistency

CSP is  $k$ -consistent if, for any set of  $k - 1$  variables and for any consistent assignment to those variables, a consistent value can always be assigned to any  $k$ -th variable

- 1-consistency: node consistency
- 2-consistency: arc consistency
- 3-consistency: path consistency

## Strong K-consistency

A CSP is strongly  $k$ -consistent if it is  $k$ -consistent and is also  $(k - 1)$ -consistent,  $(k - 2)$ -consistent,  $\dots$ , all the way down to 1-consistent.

- Time complexity for finding a solution:  $\mathcal{O}(n^2d)$  to find a solution for strongly  $n$ -consistent CSP with  $n$  nodes
- Time and space complexity for establishing  $n$ -consistency: exponential in  $n$

- **Global consistency:** One involving an arbitrary number of variables (but not necessarily all variables)
  - For example, the *Alldiff* constraint: all the variables involved must have distinct values.
- Simple method of inconsistency detection for *Alldiff* constraints: if  $m$  variables are involved in the constraint with  $n$  possible distinct values,  $m > n$ , then the constraint cannot be satisfied.
- Example: Detecting inconsistency in  $\{WA = red, NSW = red\}$ 
  - Consider  $SA$ ,  $NT$ , and  $Q$  that are effectively connected by an *Alldiff* constraint.
  - After applying AC-3 with the partial assignment, the domain of each variable is reduced to  $\{green, blue\}$ .
  - But,  $m > n$  ( $3 > 2$ ), so the *Alldiff* constraint is violated.

# Example: Sudoku

- A Sudoku puzzle and its solution

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7								8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

	1	2	3	4	5	6	7	8	9
A	4	8	3	9	2	1	6	5	7
B	9	6	7	3	4	5	8	2	1
C	2	5	1	8	7	6	4	9	3
D	5	4	8	1	3	2	9	7	6
E	7	2	9	5	6	4	1	3	8
F	1	3	6	7	9	8	2	4	5
G	3	7	2	6	8	9	5	1	4
H	8	1	4	2	5	3	7	6	9
I	6	9	5	4	1	7	3	8	2

- How far can arc consistency take us?