

# Action Planning: Recent Theoretical and Practical Advances

## Part 2: Computational Complexity

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# Action Planning – Computational Complexity

Motivation

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Propositional Planning

Domain-Dependent Planning

# How to Plan Efficiently?

- ▶ The **basic planning algorithms** presented in the last lecture seem to suggest that the planning problem is inherently difficult, i.e., requires **exponential** time.
  - ▶ Is this **really** true?
  - ▶ **How** difficult is planning for the different formalisms and approaches?
  - ▶ **Where** are the sources of difficulty?
- ↪ Determine **computational complexity** and **decidability**.

# Motivation: Why Complexity Analysis?

- ▶ Why care about **computational complexity**?
  - ▶ Planning is so difficult that it does not make much sense to try to design algorithms that are efficient in general
  - ▶ The **worst case complexity** does not matter – write fast algorithms for the problems that come up
- ~> All true, but . . .
- ▶ Try to avoid backtracking/search algorithms if problems are **simple**
  - ▶ Find out whether methods for NP-complete problems are **applicable** (local search, backtracking, . . .)
  - ▶ Determine complexity of supposedly easier sub-problems that can be used in solving the original problem (either as a **heuristic** or as a **sub-problem** in backtracking search)

# Familiar Notions and Terminology

I assume that you are *familiar* with the following notions:

- ▶ Turing Machine
- ▶ Decision problem
- ▶ Problem and problem instance
- ▶ Decidability and undecidability
- ▶ Complexity class
- ▶ P and NP, as well as NP-hard and NP-complete
- ▶ Polynomial time reduction

# Problems, Solutions, and Complexity

- ▶ A **problem** is a set of pairs  $(I, A)$  of strings in  $\{0, 1\}^*$ .

$I$ : Instance

$A$ : Answer. If  $A \in \{0, 1\}$ : **decision problem**

↪ A **decision problem** is the same as a **formal language** (namely a set of strings)

- ▶ An algorithm **decides** (or solves) a problem if it computes the right answer for all instances.
- ▶ The **complexity of an algorithm** is a function

$$T: \mathbf{N} \rightarrow \mathbf{N},$$

measuring the *number of basic steps* (or memory requirement) the algorithm needs to compute an answer depending on the *size* of the instance.

- ▶ The **complexity of a problem** is the complexity of the most efficient algorithm that solves this problem.

# Complexity Classes

Problems are categorized into **complexity classes** according to the requirements of computational resources:

- ▶ The class of problems decidable on **deterministic Turing machines** in **polynomial time**: **P**
- Problems in **P** are said to be **efficiently solvable** (although this might not be true if the exponent is very large)
- ↪ In practice, this notion appears to more **often reasonable** than not
- ▶ The class of problems decidable on **nondeterministic Turing machines** in **polynomial time**: **NP**
- ▶ More classes, such as PSPACE, EXPTIME, etc that are defined by other resource bounds on time and memory.

# Upper and Lower Bounds

- ▶ **Upper bounds** (**membership** in a class) are usually easy to prove: Provide an **algorithm** and show that the resource bounds are respected.
  - ▶ **Lower bounds** (**hardness** for a class) are usually difficult to show.
- ↪ The technical tool here is the **polynomial reduction**.
- ▶ Given two languages  $L_1$  and  $L_2$ ,  $L_1$  can be *polynomially reduced to*  $L_2$ , written  $L_1 \leq_p L_2$ , iff there exists a polynomially computable function  $f$  such that

$$x \in L_1 \text{ iff } f(x) \in L_2$$

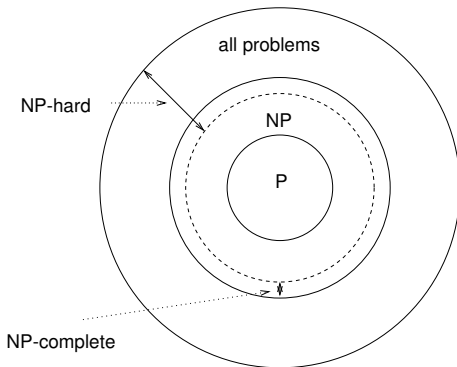
→ It cannot be harder to decide  $L_1$  than  $L_2$

- ▶  $L$  is **hard** for a class  $\mathbf{X}$  iff all languages of this class reduce to it.



# NP-complete Problems

- ▶ A problem is **NP-complete** iff it is **NP-hard** and **in NP**.
- ▶ Example: **SAT** – the satisfiability problem for propositional logic – is NP-complete (Cook/Karp)



# Beyond NP

There is more than P and NP ...

## Definition (**PSPACE**)

**PSPACE** (**NPSPACE**) is the class of decision problems that can be decided on **deterministic** (**non-deterministic**) Turing machines using only **polynomial many tape cells**.

Some facts about PSPACE:

- ▶ PSPACE is **closed under complements** (as all other deterministic classes)
- ▶ PSPACE is **identical** to NPSPACE (because non-deterministic Turing machines can be simulated on deterministic TMs using only quadratic space)
- ▶  $NP \subseteq PSPACE$  (because in polynomial time one can “visit” only polynomial space, i.e.,  $NP \subseteq NPSPACE$ )
- ▶ It is **unknown** whether  $NP \neq PSPACE$ , but it is **believed** that this is true.

# PSPACE-completeness

## Definition (**PSPACE-completeness**)

A decision problem (or language) is **PSPACE-complete**, if it is in PSPACE and all other problems in PSPACE can be polynomially reduced to it.

Intuitively, **PSPACE-complete** problems are the “hardest” problems in PSPACE (similar to NP-completeness). They appear to be “harder” than **NP-complete** problems from a *practical point of view*.

An example for a PSPACE-complete problem is the

*NFA equivalence problem*:

**Instance:** Two non-deterministic finite state automata  $A_1$  and  $A_2$ .

**Question:** Are the languages accepted by  $A_1$  and  $A_2$  identical?

## Other Complexity Classes ...

- ▶ There are complexity classes **above PSPACE** (EXPTIME, EXPSPACE, NEXPTIME, DEXPTIME, ...),
  - ▶ there are (infinitely many) classes **between NP and PSPACE** (the polynomial hierarchy defined by **oracle machines**)
  - ▶ there are (infinitely many) classes **inside P** (circuit classes with different depths)
- and for most of the classes *we do not know* whether the containment relationships are **strict**

# The Planning Problem – Formally

Definition (Plan existence problem (**PLANEX**))

**Instance:**  $\Pi = \langle \mathcal{S}, \mathbf{O}, \mathbf{I}, \mathbf{G} \rangle$ .

**Question:** Does there exist a plan  $\Delta$  that solves  $\Pi$ , i.e.,  $Res(\mathbf{I}, \Delta) \supseteq \mathbf{G}$ ?

Definition (Bounded plan existence problem (**PLANLEN**))

**Instance:**  $\Pi = \langle \mathcal{S}, \mathbf{O}, \mathbf{I}, \mathbf{G} \rangle$  and a positive integer  $n$ .

**Question:** Does there exist a plan  $\Delta$  of length  $n$  or less that solves  $\Pi$ ?

From a practical point of view, also **PLANGEN** (*generating* a plan that solves  $\Pi$ ) and **PLANLENGEN** (*generating* a plan of length  $n$  that solves  $\Pi$ ) and **PLANOPT** (*generating* an optimal plan) are interesting (but at least as hard as the decision problems).

## Basic STRIPS with First-Order Terms

- ▶ The state space for STRIPS with general first-order terms is **infinite**
  - ▶ We can use function terms to describe (the index of) *tape cells of a Turing machine*
  - ▶ We can use operators to describe the *Turing machine control*
  - ▶ The existence of a plan is then equivalent to the existence of a *successful computation* on the Turing machine
- ↪ PLANEX for STRIPS with first-order terms can be used to decide the **Halting problem**

### Theorem

*PLANEX for STRIPS with first-order terms is **undecidable**.*

# Turing Machines

**Deterministic Turing machine**  $M = \langle Q, \Sigma, \Gamma, \delta, \#, q_0, F \rangle$ :

- ▶  $Q$ : finite set of states
- ▶  $\Sigma$ : finite input alphabet
- ▶  $\Gamma \supseteq \Sigma$ : finite tape alphabet
- ▶  $\delta: Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$  transition function
- ▶  $\# \in \Sigma$ : blank symbol
- ▶  $q_0 \in Q$ : initial state
- ▶  $F \subseteq Q$ : terminal states

# Reduction

Let  $w \in \Sigma^*$  be an *input string* and let  $M = \langle Q, \Sigma, \Gamma, \delta, \#, q_0, F \rangle$  be a *deterministic Turing machine*  $\rightsquigarrow$  *planning instance*  $\Pi = \langle \mathcal{S}, \mathbf{O}, \mathbf{I}, \mathbf{G} \rangle$ :

- ▶ Let the *signature* be  $\mathcal{S} = \langle \{\mathbf{s}(\cdot)\}, Q \cup \Gamma \cup \{\#\}, \{\text{acc}, \text{in}(\cdot, \cdot), \text{last}(\cdot), \text{at}(\cdot, \cdot)\} \rangle$
- ▶ For given *input string*  $w = a_1 \dots a_n$ , the *initial state* and *goal specification* is

$$\begin{aligned} \mathbf{I} &= \{ \text{in}(\mathbf{s}(\#), a_1), \dots, \text{in}(\mathbf{s}^n(\#), a_n), \text{last}(\mathbf{s}^n(\#)), \text{at}(\mathbf{s}(\#), q_0) \}, \\ \mathbf{G} &= \{ \text{acc} \}. \end{aligned}$$

- ▶ For each *transition* of the form  $\delta(q, a) = \langle q', a', R \rangle$  the following *operator* is included (similar for left moves!)

$$\langle \langle X, Y \rangle, \{ \text{at}(X, q), \text{in}(X, a), \text{in}(\mathbf{s}(X), Y) \}, \{ \text{at}(\mathbf{s}(X), q'), \text{in}(X, a'), \neg \text{at}(X, q), \neg \text{in}(X, a) \} \rangle.$$

- ▶ *Extending the tape* on the right side:

$$\langle \langle X \rangle, \{ \text{last}(X) \}, \{ \text{in}(\mathbf{s}(X), \#), \text{last}(\mathbf{s}(X)), \neg \text{last}(X) \} \rangle.$$

- ▶ For each *terminal state*  $q_f \in F$ :  $\langle \langle X \rangle, \{ \text{at}(X, q_f) \}, \{ \text{acc} \} \rangle.$



# Simplifications

- ▶ Is STRIPS with only positive effects (*monotonic STRIPS*) decidable?
- ↪ Reduction from logic programming: **Still undecidable!**
- ▶ Is *function-free STRIPS* decidable?
- ↪ Finite state space, but **EXSPACE-complete** (idea: generic reduction, use a counter to initialize tape)
- ▶ What about *propositional STRIPS*?

# Propositional STRIPS

## Theorem

*PLANEX is **PSPACE-complete** for propositional STRIPS.*

- Membership follows because we can successively guess operators and compute the resulting states (needs only polynomial space)
- Hardness follows using again a **generic reduction** from TM acceptance. Instantiate polynomially many tape cells with no possibility to extend the tape (only poly. space, can all be generated in poly. time)
- ▶ PLANLEN is also PSPACE-complete (membership is easy, hardness follows by setting  $k = 2^{|\Sigma|}$ )

# Restrictions on Plans

- ▶ If we restrict the length of the plans to be only **polynomial** in the size of the planning task, **PLANEX** becomes **NP-complete**
- ▶ Similarly, if we use a **unary** representation of the natural number  $k$ , then **PLANLEN** becomes **NP-complete**
- Membership obvious (guess & check)
- Hardness by a straightforward reduction from SAT or by a generic reduction.
- ↪ One source of complexity in planning stems from the fact that plans can become **very long**
- ↪ We are only interested in short plans!
- ↪ We can use methods for NP-complete problems if we are only looking for “short” plans.

# Propositional, Precondition-free STRIPS<sub>N</sub>

In the following, we will consider STRIPS with negative preconditions:

STRIPS<sub>N</sub>

## Theorem

*The problem of deciding plan existence for precondition-free, propositional STRIPS<sub>N</sub> is in P.*

## Proof.

Do a backward greedy plan generation. Choose all operators that make some goals true and that do not make any goals false. Remove the satisfied goals and the operators from further consideration and iterate the step. Continue until all remaining goals are satisfied by the initial state (**succeed**) or no more operators can be applied (**fail**). □

# Propositional, Precondition-free STRIPS<sub>N</sub> and Plan Length

## Theorem

*The problem of deciding whether there exists a plan of length  $k$  for precondition-free, propositional STRIPS<sub>N</sub> is NP-complete, even if all effects are positive.*

## Proof.

Membership in NP is obvious. Hardness follows from a straightforward reduction from the MINIMUM-COVER problem [Garey & Johnson 79]:

Given a collection  $C$  of subsets of a finite set  $S$  and a positive integer  $k$ , does there exist a cover for  $S$  of size  $k$  or less, i.e., a subset  $C' \subseteq C$  such that  $\bigcup C' \supseteq S$  and  $|C'| \leq k$ ?



We will use this result later

# NP-complete Planning: One Precondition and one Positive Effect

## Theorem

*The problem of deciding plan existence if all operators have at most one precondition and one positive effect is NP-complete.*

## Proof.

**Membership** in NP follows because all effects are positive so that an operator should be applied only once  $\rightsquigarrow$  easy guess & check in poly. time.

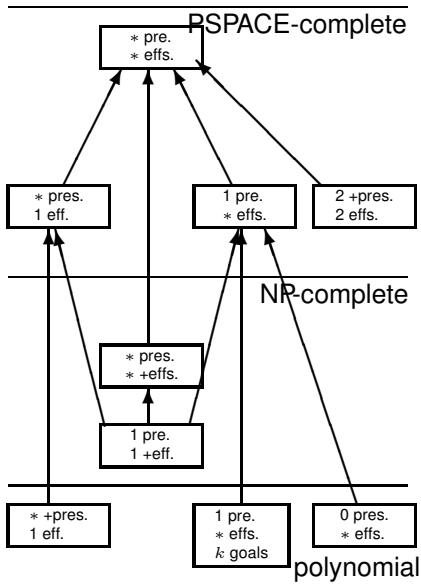
**Hardness** follows by a straightforward reduction from SAT (assuming CNF):

- ▶ Propositional atoms in the planning instance:  $T_i$  (the  $i$ th atom is true),  $F_i$  (... false), and  $C_j$  ( $j$ th clause is true)
  - ▶ Initial state  $\mathbf{I} = \emptyset$ , goal  $\mathbf{G} = \{C_1, \dots, C_m\}$
  - ▶ Truth value choosers:  $\langle \{\neg F_i\}, \{T_i\} \rangle, \langle \{\neg T_i\}, \{F_i\} \rangle$ .
  - ▶ Literal evaluator (positive occurrence of  $i$ th atom in  $j$ th clause):  $\langle \{T_i\}, \{C_j\} \rangle$
  - ▶ Negative occurrence:  $\langle \{F_i\}, \{C_j\} \rangle$
- $\rightsquigarrow$  There exists a (poly size) plan iff the formula is satisfiable.



# More Results for Specializations of STRIPS<sub>N</sub>

- ▶  $n$  pres./effs.  
means that the number of *preconditions* resp. *effect* is restricted to  $n$  (“\*” means no restriction)
- ▶ +pres./+effs.  
means only positive *preconditions* resp. *effects*
- ▶  $k$  goals means that at most  $k$  *goals* are allowed



# Domain-Dependent Planning

Planning (and its complexity) for particular domains is interesting, since we want to **judge** planning benchmarks. . . and perhaps want to go for **domain-dependent planning**.

## Theorem

*DATALOG-STRIPS-PLANEX with a fixed set of operators is PSPACE-complete.*

## Proof.

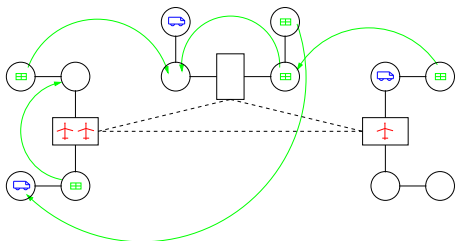
Membership follows because the arity of predicates in the operator is fixed in advance, i.e., we have to generate only polynomially many (relevant) operators and ground atoms for a given set of constants. Hence, planning can be reduced to the propositional case.

Hardness follows for the fixed set of operators for a fixed PSPACE-complete language constructed in the PSPACE-completeness proof. □



## A Concrete Domain: Logistics

There are several cities, each containing several locations, some of which are airports. There are also trucks, which drive within a single city, and airplane, which can fly between airports. The goal is to get some packages from various locations to various new locations [McDermott, 1998].



# Plan Existence for Logistics

## Theorem

*PLANEX for Logistics can be decided in polynomial time*

## Proof.

Consider the subgraphs formed by the **connected airport networks** for planes and the **city networks** for trucks. If at least one vehicle (truck or plane) is in one of the subgraphs, all nodes in the subgraph are internally reachable, otherwise only the externally connected nodes can be reached. Check for each package delivery, whether there are connected subgraphs such that the package can pass through the subgraphs to the target node. This is a simple reachability test, which can be done in **poly. time**. □

# Optimizing Delivery: Shortest Plans for Logistics

## Definition (Feedback Vertex Set)

- ▶ Given: a directed graph  $G=(V,A)$  and a natural number  $k$
- ▶ Question: Does there exist a subset  $V' \subseteq V$  such that removing  $V'$  results in an acyclic graph?

This problem is NP-complete and can be used to prove the following result:

## Theorem

*PLANLEN for Logistics is NP-complete, even if there is only one complete city graph and one truck in this graph.*

## Proof.

Membership follows because there is an obvious polynomial upper bound of moves for all solvable instances.

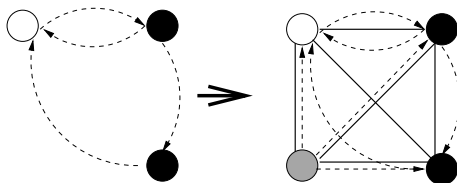
Hardness is shown using a reduction from **FVS**.

# PLANLEN for Logistics: NP-hardness contd. (1)

## Proof. (continued)

Let  $G = (V, A)$  a directed graph and  $k$  a natural number. Then  $G$  contains a FVS iff the logistics problem constructed below has a plan of length at most  $3|V| + 2|A| + k$ .

Construct a *Logistics* task with just one city network which is a complete graph containing  $V$  and an extra start node  $v_0$ . The truck has to deliver one package from  $v_0$  to each other location and one package from  $u$  to  $v$  for each  $(u, v) \in V$ .



## PLANLEN for Logistics: NP-hardness contd. (2)

### Proof. (continued).

Let  $V' \subseteq V$  the feedback vertex set. Solve task by moving to  $V'$  in any order, then to all  $V - V'$  using a topological ordering on these nodes, and finally to  $V'$  again. Requires  $|A| + |V|$  **load** and **unload** actions each, and  $|V'| + |V - V'| + |V'|$  **movements**, i.e.,  $3|V| + 2|A| + k$  **actions** if  $|V'| \leq k$ .

Conversely, at least  $3|V| + 2|A|$  actions are needed. If a plan contains not more than  $3|V| + 2|A| + k$ , then no more than  $k$  nodes are visited twice. These nodes form a **FVS**. □

**Note:** It is not route planning that makes the task difficult, but the interaction of sub-goals!






## Other Domains

- ▶ Helmert has analyzed (generalizations) of all other domains that have been used at the [international planning competition](#)
- ▶ Many show a similar behavior as *Logistics*: PLANEX is in **P**, PLANLEN is **NP-complete**, e.g., *Block-World*
- ▶ Some are already NP-complete for PLANEX, e.g. *Freecell*
- ▶ No domain is **PSPACE-complete**, though (which could only happen if plans do not have polynomial length upper bound)

# Summary

- ▶ Planning with **STRIPS** using **first-order** terms is **undecidable**
- ▶ Planning with **DATALOG-STRIPS** (function free) is **EXPSPACE-complete**
- ▶ Planning with **propositional STRIPS** is **PSPACE-complete**
- ▶ If consider only “short” plans, the complexity comes down to **NP-completeness**
- ▶ **Domain dependent** planning can be easier
- ▶ **DATALOG-STRIPS** for an arbitrary fixed domain is **PSPACE-complete**
- ▶ For *Logistics*, the existence problem is in **P**, while the optimization problem is **NP-complete**, which holds for many other domains as well

# Literature

-  C. H. Papadimitriou, *Computational Complexity*, Reading, MA: Addison-Wesley, 1994.
-  T. Bylander, The Computational Complexity of Propositional STRIPS Planning, *Artificial Intelligence* **69** (1–2): 165–204, 1994.
-  K. Erol, D. S. Nau, V. S. Subrahmanian, Complexity, decidability and undecidability results for domain-independent planning, *Artificial Intelligence* **76** (1–2): 75–88, 1995.
-  N. Gupta and D. S. Nau, On the Complexity of Blocks-World Planning, *Artificial Intelligence* **56** (2): 223–254, 1992.
-  M. Helmert, Complexity results for standard benchmark domains in planning, *Artificial Intelligence* **143** (2): 219–262, 2003.