

At the end of the class you should be able to:

- recognize and represent constraint satisfaction problems
- show how constraint satisfaction problems can be solved with search
- implement and trace arc-consistency of a constraint graph
- show how domain splitting can solve constraint problems

Posing a Constraint Satisfaction Problem

A CSP is characterized by

- A set of variables V_1, V_2, \dots, V_n .
- Each variable V_i has an associated domain \mathbf{D}_{V_i} of possible values.
- There are hard constraints on various subsets of the variables which specify legal combinations of values for these variables.
- A solution to the CSP is an assignment of a value to each variable that satisfies all the constraints.

Example: scheduling activities

- **Variables:** A, B, C, D, E that represent the starting times of various activities.
- **Domains:** $\mathbf{D}_A = \{1, 2, 3, 4\}$, $\mathbf{D}_B = \{1, 2, 3, 4\}$,
 $\mathbf{D}_C = \{1, 2, 3, 4\}$, $\mathbf{D}_D = \{1, 2, 3, 4\}$, $\mathbf{D}_E = \{1, 2, 3, 4\}$
- **Constraints:**

$$(B \neq 3) \wedge (C \neq 2) \wedge (A \neq B) \wedge (B \neq C) \wedge \\ (C < D) \wedge (A = D) \wedge (E < A) \wedge (E < B) \wedge \\ (E < C) \wedge (E < D) \wedge (B \neq D).$$

Generate-and-Test Algorithm

- Generate the assignment space $\mathbf{D} = \mathbf{D}_{V_1} \times \mathbf{D}_{V_2} \times \dots \times \mathbf{D}_{V_n}$.
Test each assignment with the constraints.

- Example:

$$\begin{aligned}\mathbf{D} &= \mathbf{D}_A \times \mathbf{D}_B \times \mathbf{D}_C \times \mathbf{D}_D \times \mathbf{D}_E \\ &= \{1, 2, 3, 4\} \times \{1, 2, 3, 4\} \times \{1, 2, 3, 4\} \\ &\quad \times \{1, 2, 3, 4\} \times \{1, 2, 3, 4\} \\ &= \{\langle 1, 1, 1, 1, 1 \rangle, \langle 1, 1, 1, 1, 2 \rangle, \dots, \langle 4, 4, 4, 4, 4 \rangle\}.\end{aligned}$$

- How many assignments need to be tested for n variables each with domain size d ?

Backtracking Algorithms

- Systematically explore \mathbf{D} by instantiating the variables one at a time
- evaluate each constraint predicate as soon as all its variables are bound
- any partial assignment that doesn't satisfy the constraint can be pruned.

Example Assignment $A = 1 \wedge B = 1$ is inconsistent with constraint $A \neq B$ regardless of the value of the other variables.

A CSP can be solved by graph-searching:

- A node is an assignment values to some of the variables.
- Suppose node N is the assignment $X_1 = v_1, \dots, X_k = v_k$.
Select a variable Y that isn't assigned in N .
For each value $y_i \in \text{dom}(Y)$
 $X_1 = v_1, \dots, X_k = v_k, Y = y_i$ is a neighbour if it is consistent with the constraints.
- The start node is the empty assignment.
- A goal node is a total assignment that satisfies the constraints.

Consistency Algorithms

- Idea: prune the domains as much as possible before selecting values from them.
- A variable is **domain consistent** if no value of the domain of the node is ruled impossible by any of the constraints.
- **Example:** Is the scheduling example domain consistent?

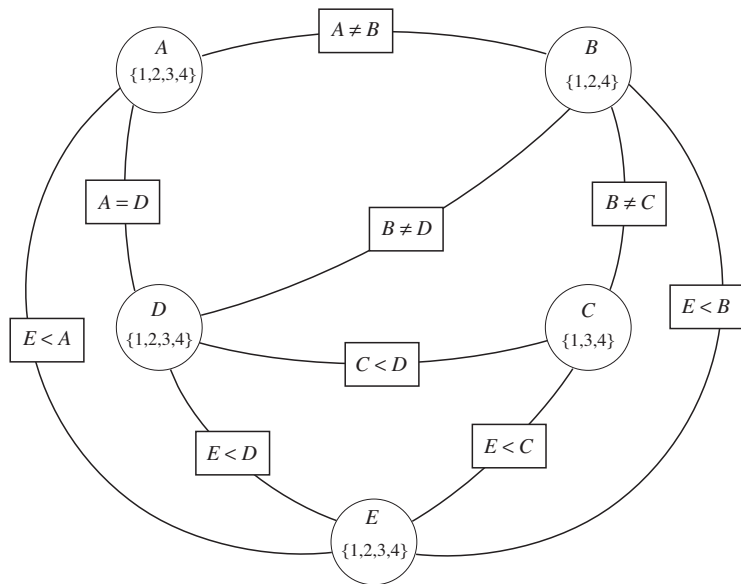
Consistency Algorithms

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- **Example:** Is the scheduling example domain consistent?
 $D_B = \{1, 2, 3, 4\}$ isn't domain consistent as $B = 3$ violates the constraint $B \neq 3$.

Constraint Network

- There is a oval-shaped node for each variable.
- There is a rectangular node for each constraint.
- There is a domain of values associated with each variable node.
- There is an arc from variable X to each constraint that involves X .

Example Constraint Network



- An arc $\langle X, r(X, \bar{Y}) \rangle$ is **arc consistent** if, for each value $x \in \text{dom}(X)$, there is some value $\bar{y} \in \text{dom}(\bar{Y})$ such that $r(x, \bar{y})$ is satisfied.
- A network is arc consistent if all its arcs are arc consistent.
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All values of X in $\text{dom}(X)$ for which there is no corresponding value in $\text{dom}(\bar{Y})$ can be deleted from $\text{dom}(X)$ to make the arc $\langle X, r(X, \bar{Y}) \rangle$ consistent.

Arc Consistency Algorithm

- The arcs can be considered in turn making each arc consistent.
- When an arc has been made arc consistent, does it ever need to be checked again?

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- Three possible outcomes when all arcs are made arc consistent: (Is there a solution?)
 - ▶ One domain is empty \implies
 - ▶ Each domain has a single value \implies
 - ▶ Some domains have more than one value \implies

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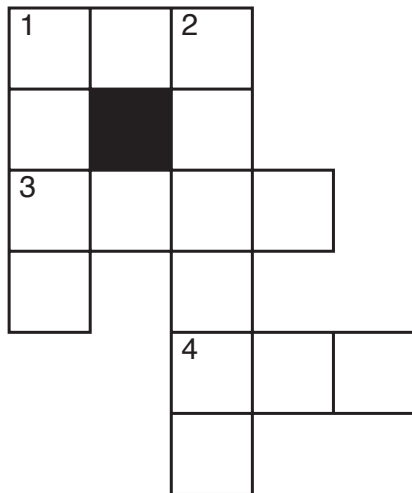
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- Three possible outcomes when all arcs are made arc consistent: (Is there a solution?)
 - ▶ One domain is empty \implies no solution
 - ▶ Each domain has a single value \implies unique solution
 - ▶ Some domains have more than one value \implies there may or may not be a solution

Finding solutions when AC finishes

- If some domains have more than one element \implies search
- Split a domain, then recursively solve each half.
- It is often best to split a domain in half.
- Do we need to restart from scratch?

Example: Crossword Puzzle



Words:

ant, big, bus, car, has
book, buys, hold,
lane, year
beast, ginger, search,
symbol, syntax

Hard and Soft Constraints

- Given a set of variables, assign a value to each variable that either
 - ▶ satisfies some set of constraints: **satisfiability problems** — “hard constraints”
 - ▶ minimizes some cost function, where each assignment of values to variables has some cost: **optimization problems** — “soft constraints”
- Many problems are a mix of hard and soft constraints (called constrained optimization problems).

Local Search (Greedy Descent):

- Maintain an assignment of a value to each variable.
- Repeat:
 - ▶ Select a variable to change
 - ▶ Select a new value for that variable
- Until a satisfying assignment is found

Local Search for CSPs

- Aim: find an assignment with zero unsatisfied constraints.
- Given an assignment of a value to each variable, a **conflict** is an unsatisfied constraint.
- The goal is an assignment with zero conflicts.
- Heuristic function to be minimized: the number of conflicts.

To choose a variable to change and a new value for it:

- Find a variable-value pair that minimizes the number of conflicts
- Select a variable that participates in the most conflicts.
Select a value that minimizes the number of conflicts.
- Select a variable that appears in any conflict.
Select a value that minimizes the number of conflicts.
- Select a variable at random.
Select a value that minimizes the number of conflicts.
- Select a variable and value at random; accept this change if it doesn't increase the number of conflicts.

- When the domains are small or unordered, the neighbors of an assignment can correspond to choosing another value for one of the variables.
- When the domains are large and ordered, the neighbors of an assignment are the adjacent values for one of the variables.
- If the domains are continuous, **Gradient descent** changes each variable proportional to the gradient of the heuristic function in that direction.

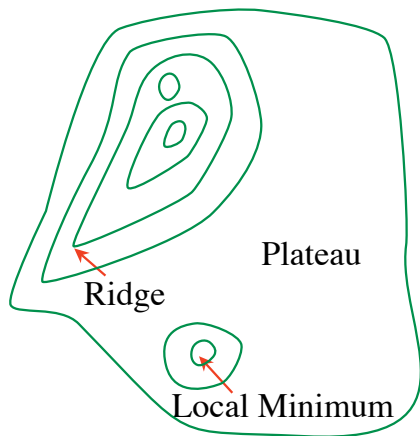
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The value of variable X_i goes from v_i to $v_i - \eta \frac{\partial h}{\partial X_i}$.
 η is the step size.

Problems with Greedy Descent

- a local minimum that is not a global minimum
- a plateau where the heuristic values are uninformative
- a ridge is a local minimum where n -step look-ahead might help



- Consider two methods to find a minimum value:
 - ▶ Greedy descent, starting from some position, keep moving down & report minimum value found
 - ▶ Pick values at random & report minimum value found
- Which do you expect to work better to find a global minimum?
- Can a mix work better?

Randomized Greedy Descent

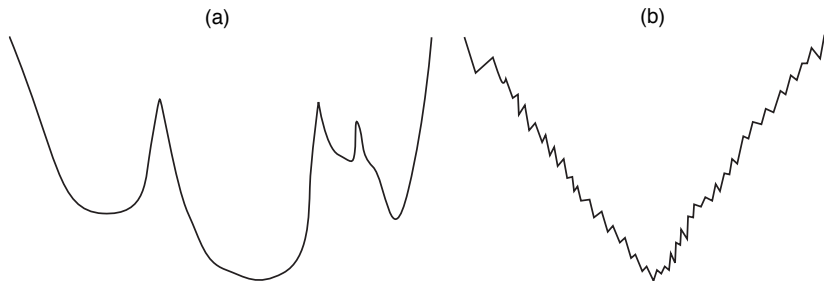
As well as downward steps we can allow for:

- **Random steps:** move to a random neighbor.
- **Random restart:** reassign random values to all variables.

Which is more expensive computationally?

1-Dimensional Ordered Examples

Two 1-dimensional search spaces; step right or left:



- Which method would most easily find the global minimum?
- What happens in hundreds or thousands of dimensions?
- What if different parts of the search space have different structure?

Stochastic local search is a mix of:

- Greedy descent: move to a lowest neighbor
- Random walk: taking some random steps
- Random restart: reassigning values to all variables

Variants of random walk:

- When choosing the best variable-value pair, randomly sometimes choose a random variable-value pair.
- When selecting a variable then a value:
 - ▶ Sometimes choose any variable that participates in the most conflicts.
 - ▶ Sometimes choose any variable that participates in any conflict (a red node).
 - ▶ Sometimes choose any variable.
- Sometimes choose the best value and sometimes choose a random value.

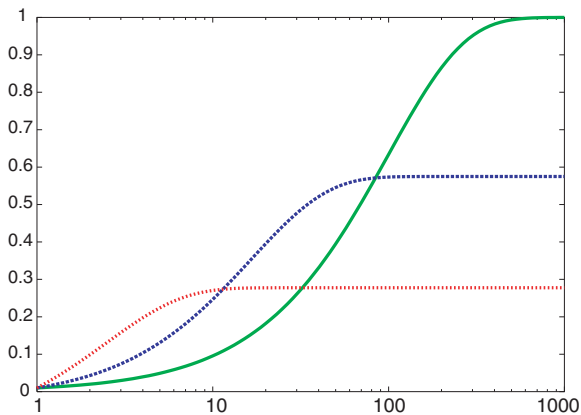
- How can you compare three algorithms when
 - ▶ one solves the problem 30% of the time very quickly but doesn't halt for the other 70% of the cases
 - ▶ one solves 60% of the cases reasonably quickly but doesn't solve the rest
 - ▶ one solves the problem in 100% of the cases, but slowly?

Comparing Stochastic Algorithms

- How can you compare three algorithms when
 - ▶ one solves the problem 30% of the time very quickly but doesn't halt for the other 70% of the cases
 - ▶ one solves 60% of the cases reasonably quickly but doesn't solve the rest
 - ▶ one solves the problem in 100% of the cases, but slowly?
- Summary statistics, such as mean run time, median run time, and mode run time don't make much sense.

Runtime Distribution

- Plots runtime (or number of steps) and the proportion (or number) of the runs that are solved within that runtime.



Variant: Simulated Annealing

- Pick a variable at random and a new value at random.
- If it is an improvement, adopt it.
- If it isn't an improvement, adopt it probabilistically depending on a temperature parameter, T .
 - ▶ With current assignment n and proposed assignment n' we move to n' with probability $e^{(h(n')-h(n))/T}$
- Temperature can be reduced.

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Probability of accepting a change:

Temperature	1-worse	2-worse	3-worse
10	0.91	0.81	0.74
1	0.37	0.14	0.05
0.25	0.02	0.0003	0.000006
0.1	0.00005	2×10^{-9}	9×10^{-14}

- To prevent cycling we can maintain a **tabu list** of the k last assignments.
- Don't allow an assignment that is already on the tabu list.
- If $k = 1$, we don't allow an assignment of to the same value to the variable chosen.
- We can implement it more efficiently than as a list of complete assignments.
- It can be expensive if k is large.

A total assignment is called an **individual**.

- **Idea:** maintain a population of k individuals instead of one.
- At every stage, update each individual in the population.
- Whenever an individual is a solution, it can be reported.
- Like k restarts, but uses k times the minimum number of steps.

- Like parallel search, with k individuals, but choose the k best out of all of the neighbors.
- When $k = 1$, it is greedy descent.
- When $k = \infty$, it is breadth-first search.
- The value of k lets us limit space and parallelism.

Stochastic Beam Search

- Like beam search, but it probabilistically chooses the k individuals at the next generation.
- The probability that a neighbor is chosen is proportional to its heuristic value.
- This maintains diversity amongst the individuals.
- The heuristic value reflects the fitness of the individual.
- Like asexual reproduction: each individual mutates and the fittest ones survive.

- Like stochastic beam search, but pairs of individuals are combined to create the offspring:
- For each generation:
 - ▶ Randomly choose pairs of individuals where the fittest individuals are more likely to be chosen.
 - ▶ For each pair, perform a cross-over: form two offspring each taking different parts of their parents:
 - ▶ Mutate some values.
- Stop when a solution is found.

- Given two individuals:

$$X_1 = a_1, X_2 = a_2, \dots, X_m = a_m$$

$$X_1 = b_1, X_2 = b_2, \dots, X_m = b_m$$

- Select i at random.
- Form two offspring:

$$X_1 = a_1, \dots, X_i = a_i, X_{i+1} = b_{i+1}, \dots, X_m = b_m$$

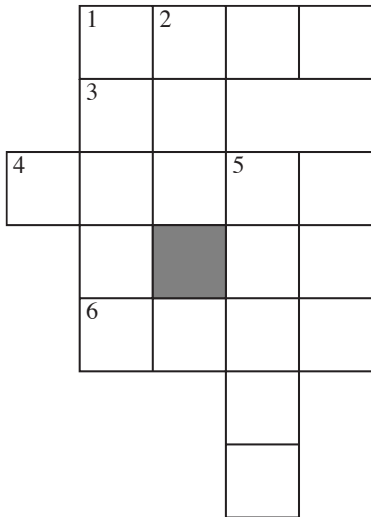
$$X_1 = b_1, \dots, X_i = b_i, X_{i+1} = a_{i+1}, \dots, X_m = a_m$$

- The effectiveness depends on the ordering of the variables.
- Many variations are possible.

Constraint satisfaction revisited

- A **Constraint Satisfaction problem** consists of:
 - ▶ a set of variables
 - ▶ a set of possible values, a **domain** for each variable
 - ▶ a set of constraints amongst subsets of the variables
- The aim is to find a set of assignments that satisfies all constraints, or to find all such assignments.

Example: crossword puzzle



at, be, he, it, on,
eta, hat, her, him,
one,
desk, dove, easy,
else, help, kind,
soon, this,
dance, first, fuels,
given, haste, loses,
sense, sound, think,
usage

Two ways to represent the crossword as a CSP

- First representation:
 - ▶ nodes represent word positions: 1-down...6-across
 - ▶ domains are the words
 - ▶ constraints specify that the letters on the intersections must be the same.
- Dual representation:
 - ▶ nodes represent the individual squares
 - ▶ domains are the letters
 - ▶ constraints specify that the words must fit

Representations for image interpretation

- First representation:
 - ▶ nodes represent the chains and regions
 - ▶ domains are the scene objects
 - ▶ constraints correspond to the intersections and adjacency
- Dual representation:
 - ▶ nodes represent the intersections
 - ▶ domains are the intersection labels
 - ▶ constraints specify that the chains must have same marking

- Idea: eliminate the variables one-by-one passing their constraints to their neighbours

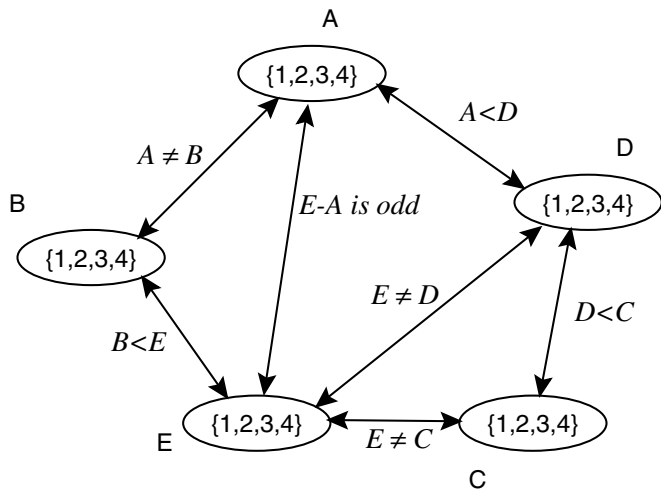
Variable Elimination Algorithm:

- If there is only one variable, return the intersection of the (unary) constraints that contain it
- Select a variable X
- Join the constraints in which X appears, forming constraint R_1
- Project R_1 onto its variables other than X , forming R_2
- Replace all of the constraints in which X_i appears by R_2
- Recursively solve the simplified problem, forming R_3
- Return R_1 joined with R_3

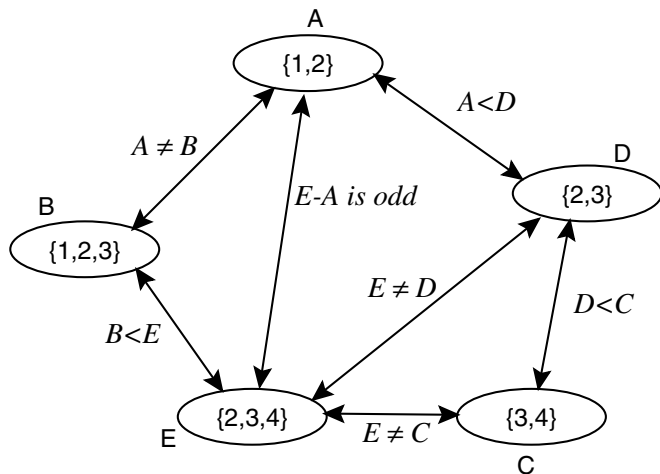
Variable elimination (cont.)

- When there is a single variable remaining, if it has no values, the network was inconsistent.
- The variables are eliminated according to some **elimination ordering**
- Different elimination orderings result in different size intermediate constraints.

Example network



Example: arc-consistent network



Example: eliminating C

$r_1 : C \neq E$	C	E
	3	2
	3	4
	4	2
	4	3

$r_2 : C > D$	C	D
	3	2
	4	2
	4	3

$r_3 : r_1 \bowtie r_2$	C	D	E
	3	2	2
	3	2	4
	4	2	2
	4	2	3
	4	3	2
	4	3	3

$r_4 : \pi_{\{D,E\}} r_3$	D	E
	2	2
	2	3
	2	4
	3	2
	3	3

 new constraint

Resulting network after eliminating C

