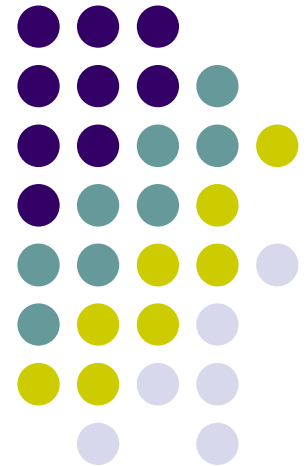


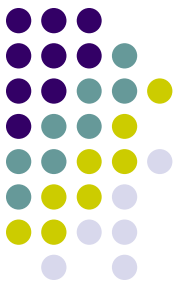
Junction tree Algorithm

10-708: Probabilistic Graphical Models

Recitation: 10/04/07

Ramesh Nallapati

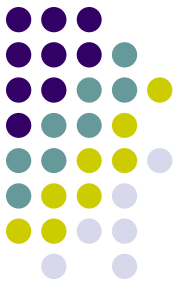




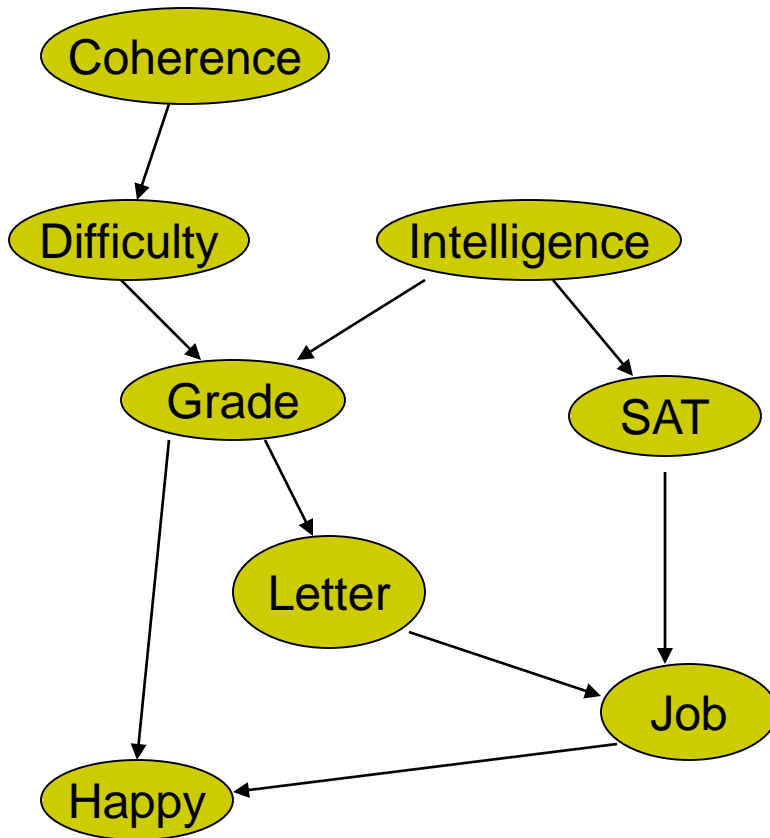
Cluster Graphs

- A **cluster graph** K for a set of factors F is an undirected graph with the following properties:
 - Each node i is associated with a subset $C_i \subset X$
 - **Family preserving property**: each factor ϕ is such that $\text{scope}[\phi] \subseteq C_i$
 - Each edge between C_i and C_j is associated with a **sepset** $S_{ij} = C_i \cap C_j$
- Execution of variable elimination defines a cluster-graph
 - Each factor used in elimination becomes a cluster-node
 - An edge is drawn between two clusters if a message is passed between them in elimination
 - Example: Next slide →

Variable Elimination to Junction Trees:



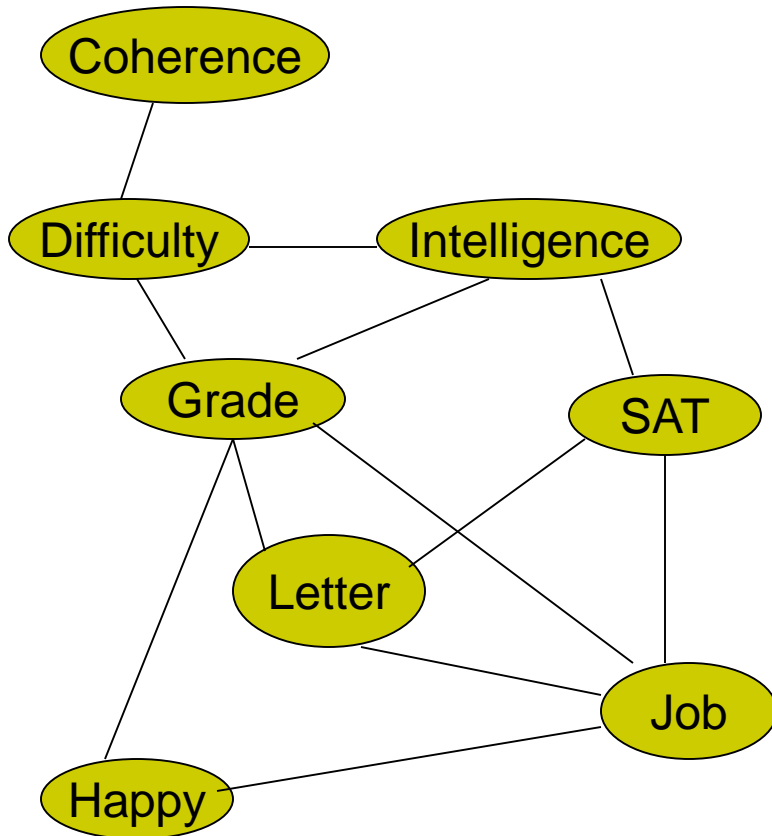
- Original graph



Variable Elimination to Junction Trees:



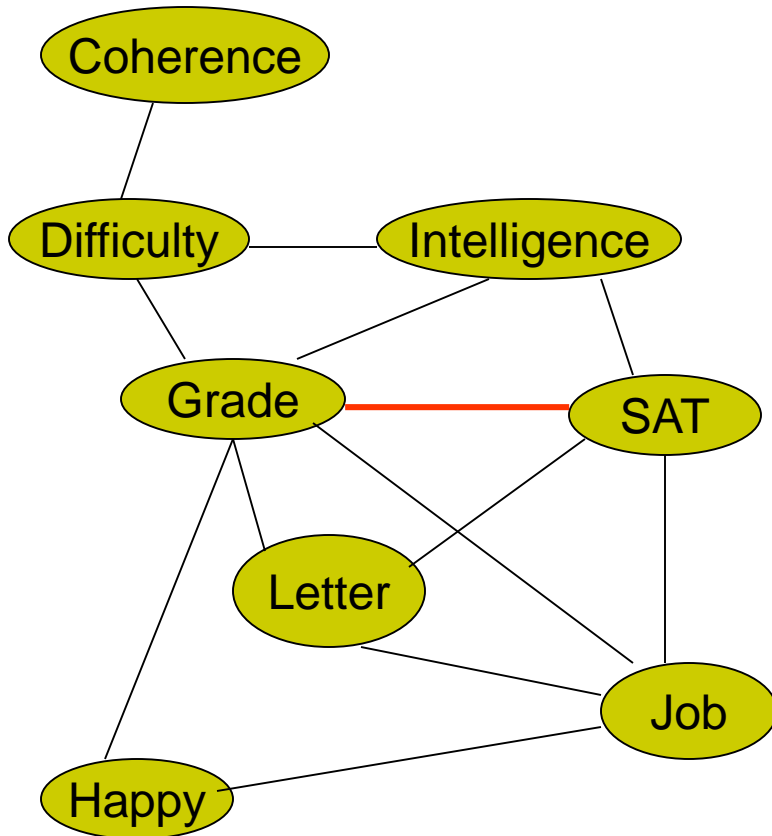
- Moralized graph



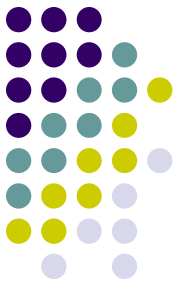
Variable Elimination to Junction Trees:



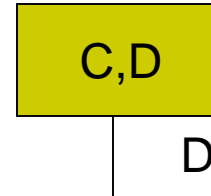
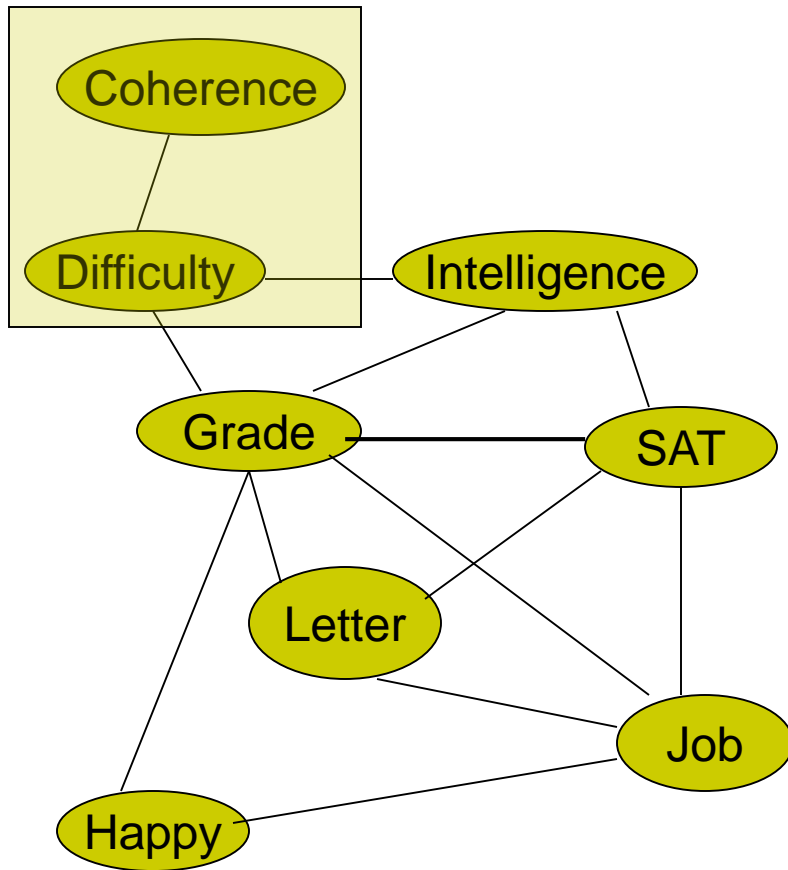
- Triangulated graph



Variable Elimination to Junction Trees:

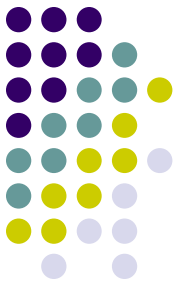


- Elimination ordering: ~~C~~, D, I, H, G, S, L

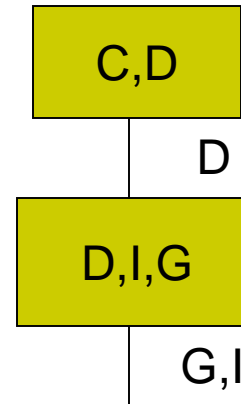
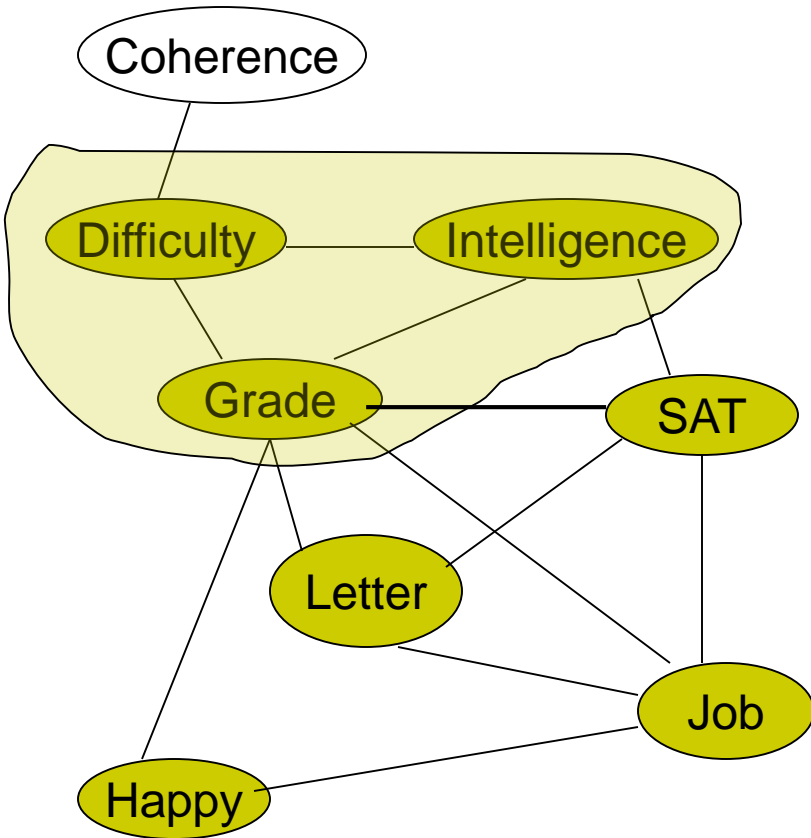


$$m_C(D) = \sum_C P(C)P(D|C)$$

Variable Elimination to Junction Trees:

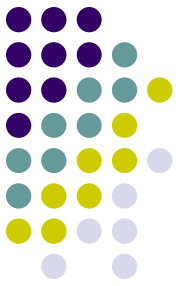


- Elimination ordering: ~~C~~, ~~D~~, I, H, G, S, L

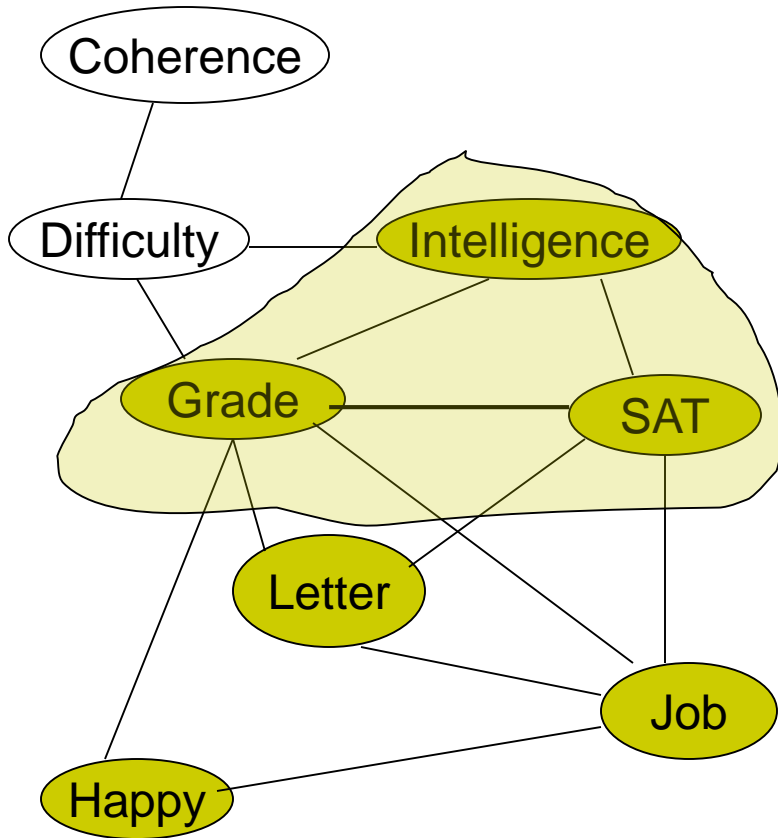


$$m_D(G, I) = \sum_D m_C(D) P(G|D, I)$$

Variable Elimination to Junction Trees:



- Elimination ordering: ~~C~~, ~~D~~, ~~I~~, H, G, S, L

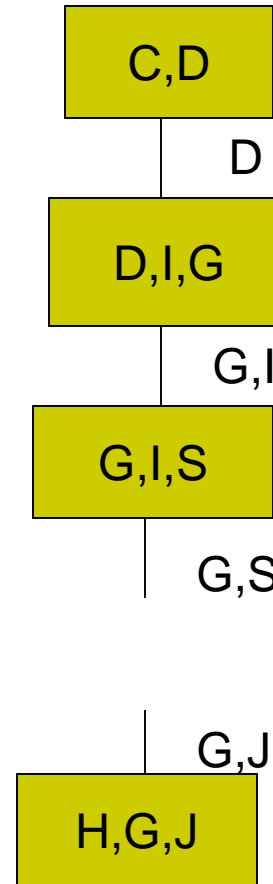
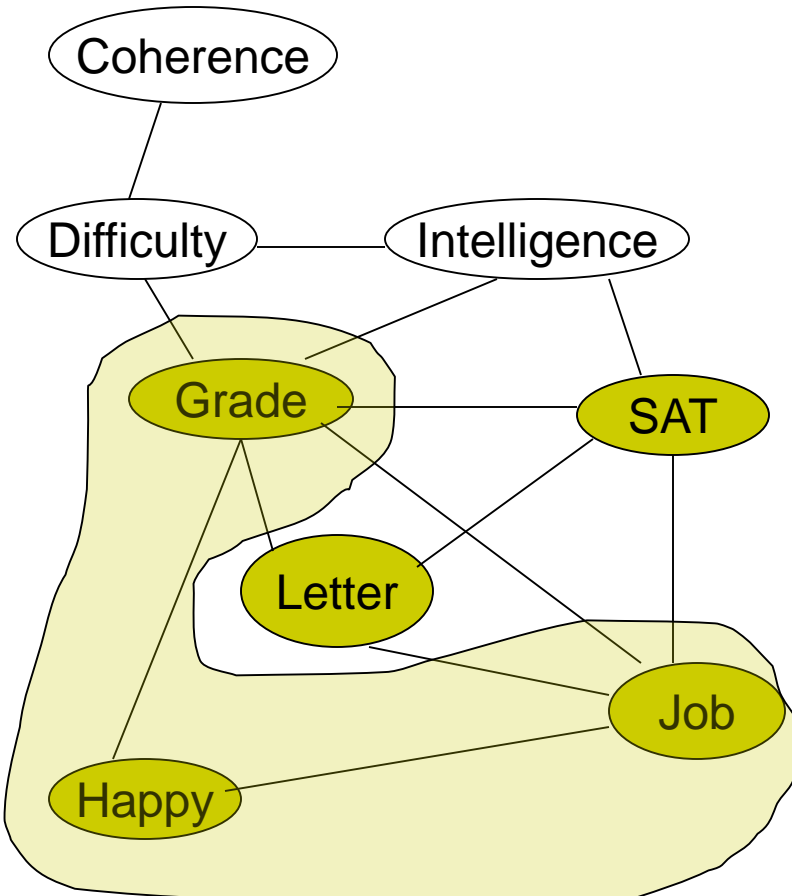


$$m_I(G, S) = \sum_I m_D(G, I) P(G|I) P(S|I)$$

Variable Elimination to Junction Trees:



- Elimination ordering: ~~C~~, ~~D~~, ~~I~~, ~~H~~, G, S, L

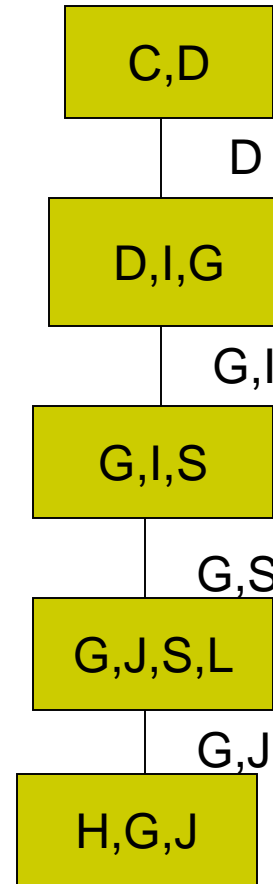
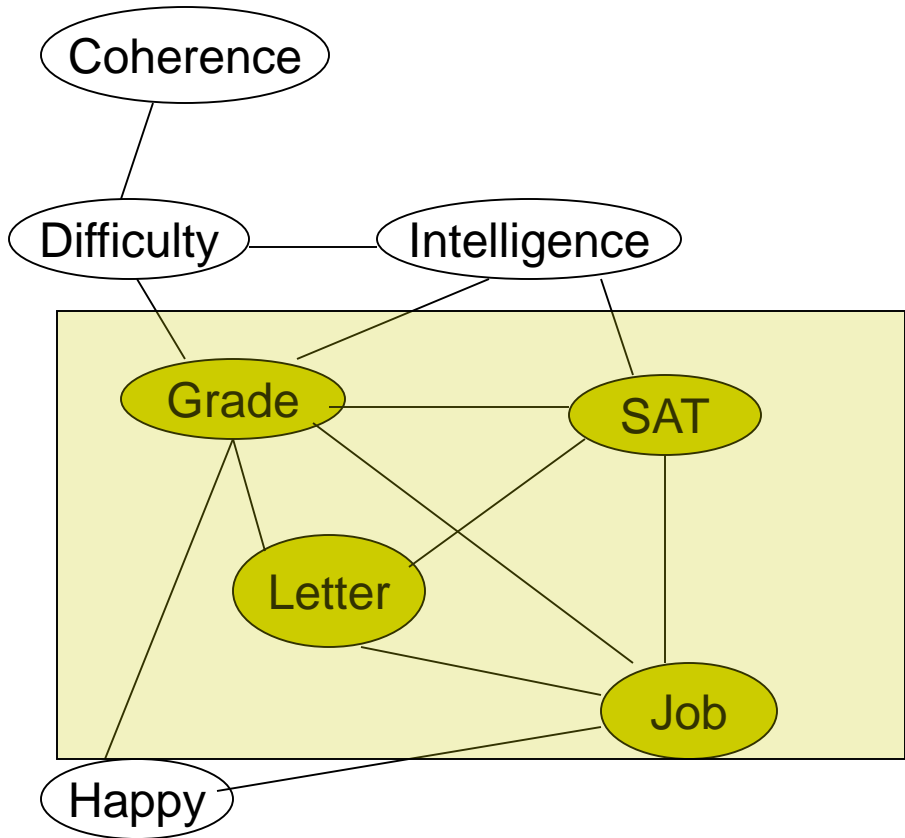


$$m_H(G, J) = \sum_H P(H|G, J)$$

Variable Elimination to Junction Trees:



- Elimination ordering: ~~C~~, ~~D~~, ~~I~~, ~~H~~, ~~G~~, S, L

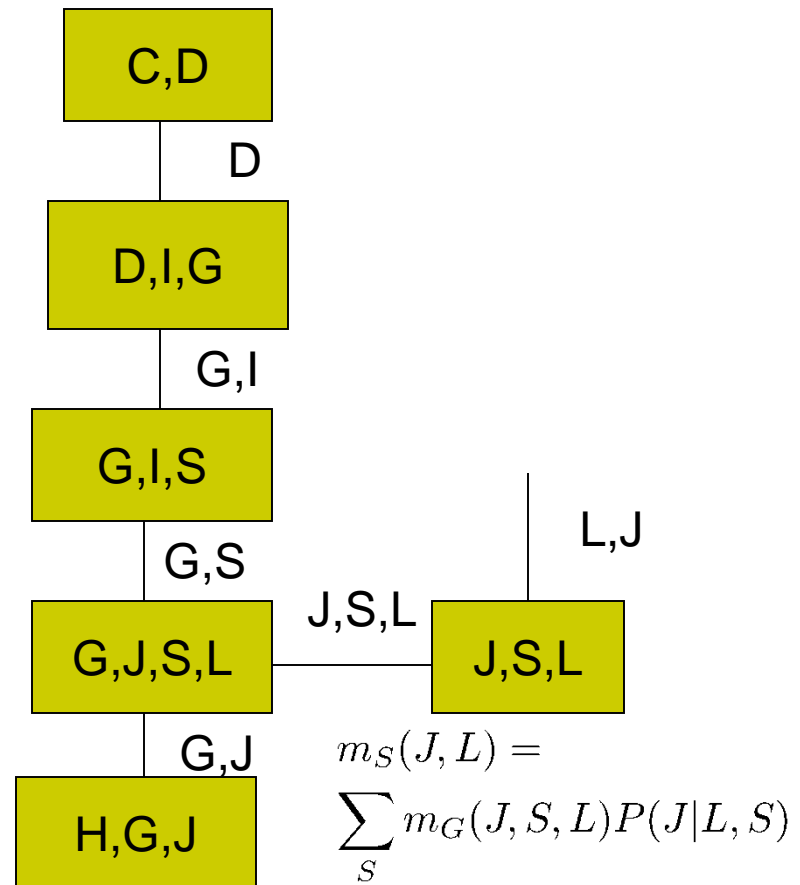
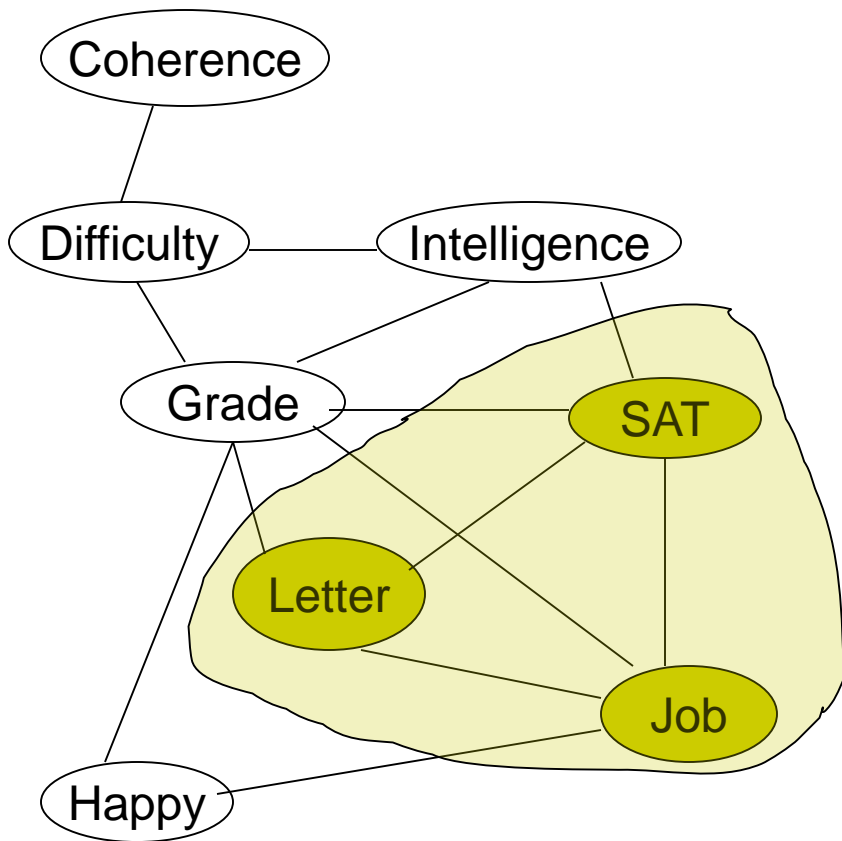


$$m_G(J, S, L) = \sum_G m_I(G, S) P(L|G) m_H(G, J)$$

Variable Elimination to Junction Trees:



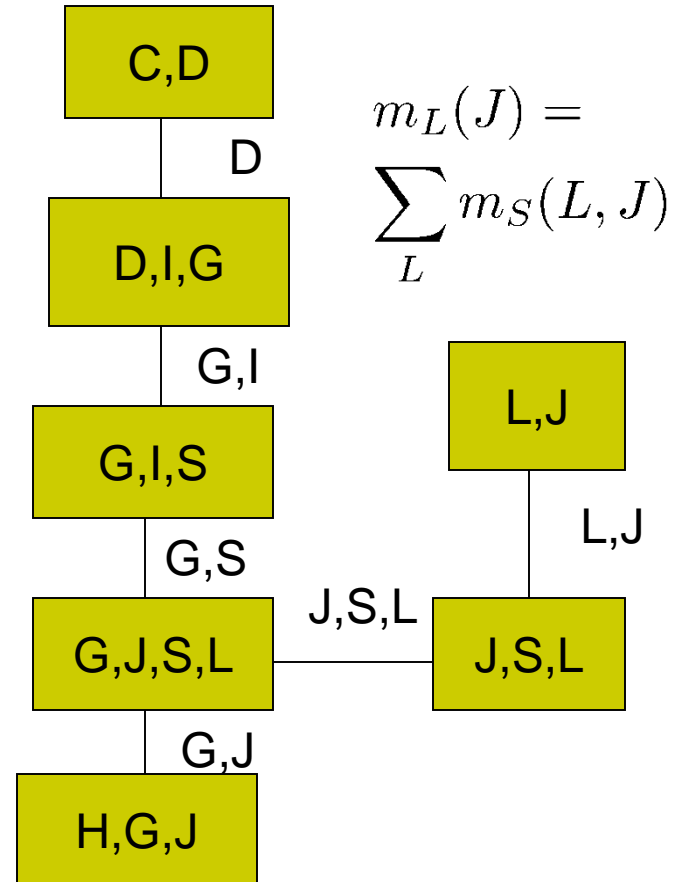
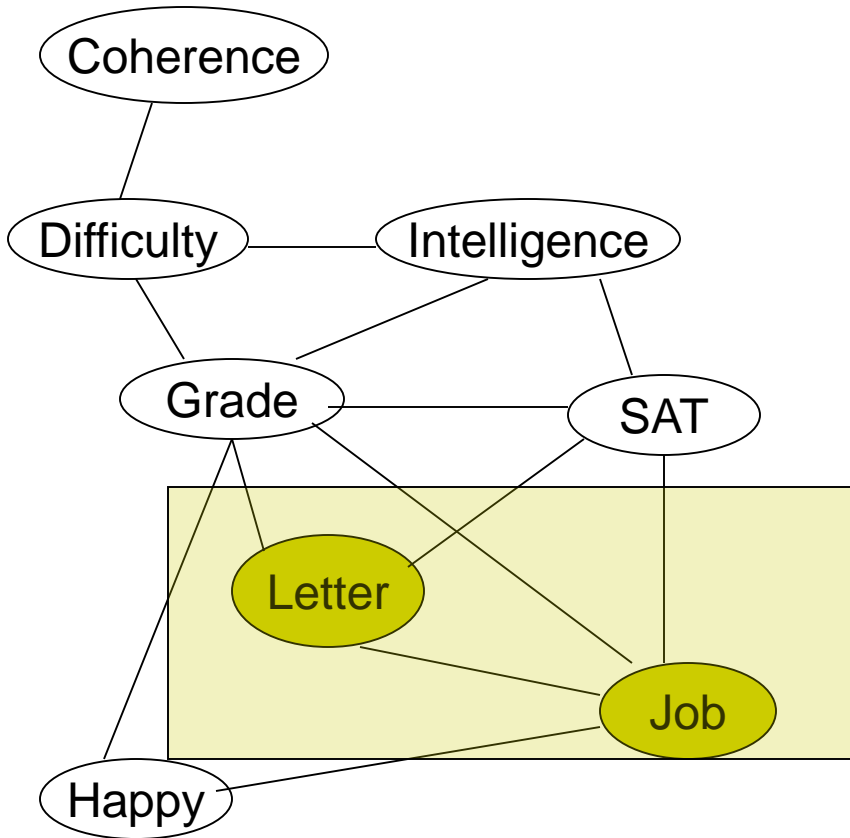
- Elimination ordering: ~~C~~, ~~D~~, ~~I~~, ~~H~~, ~~G~~, ~~S~~, L



Variable Elimination to Junction Trees:



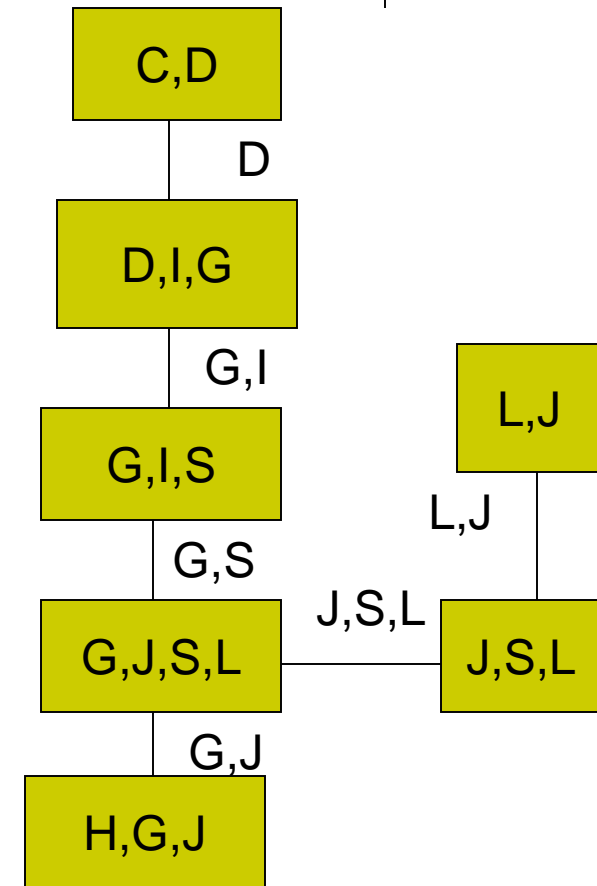
- Elimination ordering: ~~C~~, ~~D~~, ~~I~~, ~~H~~, ~~G~~, ~~S~~, ~~L~~



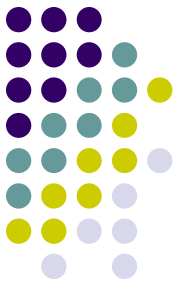
Properties of Junction Tree



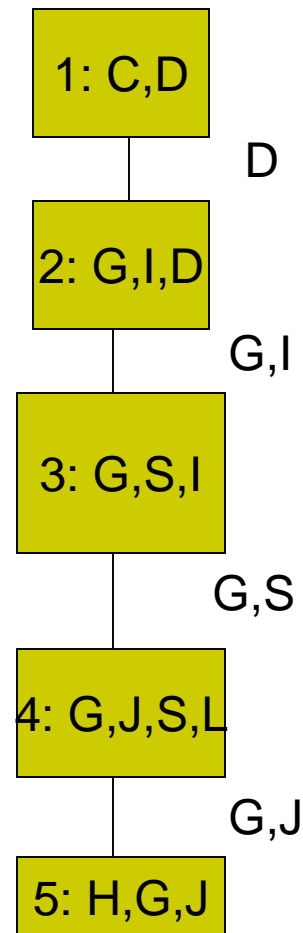
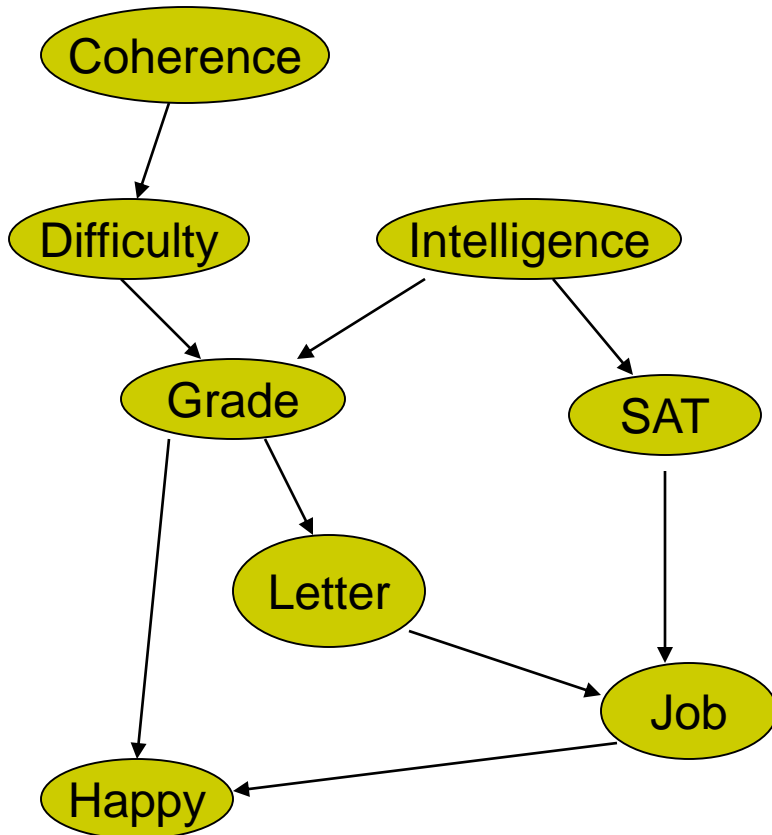
- Cluster-graph G induced by variable elimination is necessarily a **tree**
 - Reason: each intermediate factor is used at most once
- G satisfies **Running Intersection Property (RIP)**
 - $(X \in C_i \ \& \ X \in C_j) \Rightarrow X \in C_k$ where C_k is in the path of C_i and C_j
- If C_i and C_j are neighboring clusters, and C_i passes message m_{ij} to C_j , then $\text{scope}[m_{ij}] = S_{i,j}$
- Let F be set of factors over X . A cluster tree over F that satisfies **RIP** is called a **junction tree**
- One can obtain a minimal junction tree by eliminating the sub-cliques
 - No redundancies



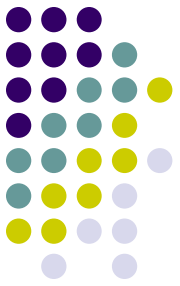
Junction Trees to Variable elimination:



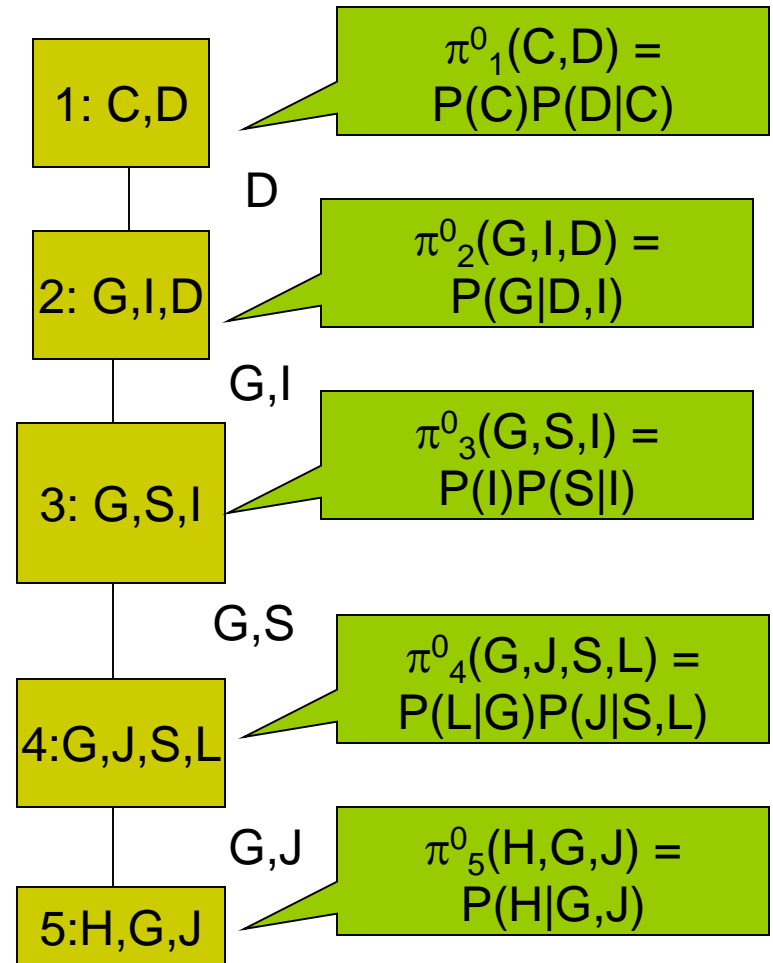
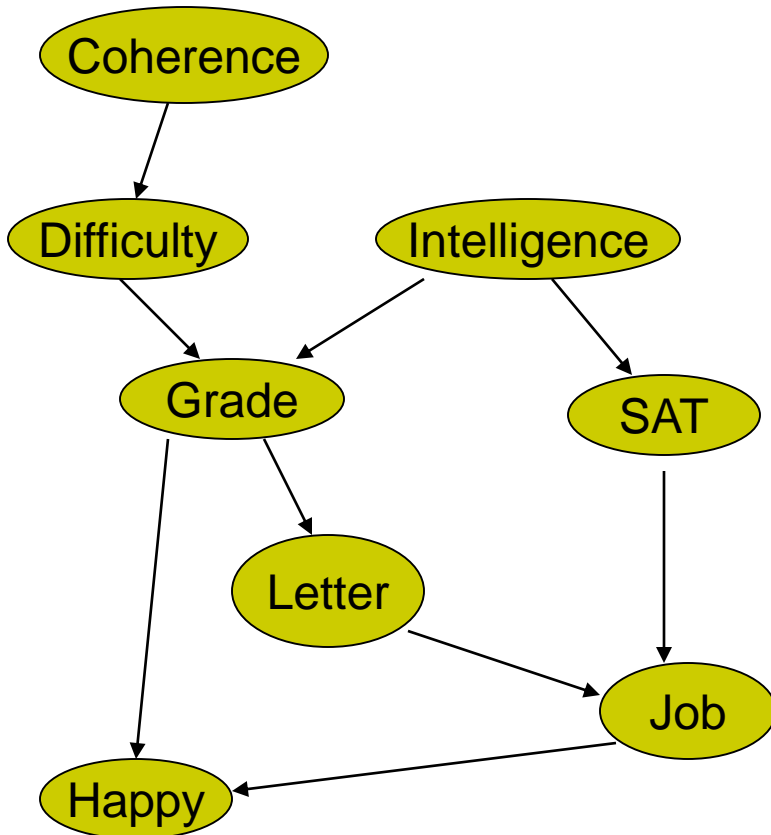
- Now we will assume a junction tree and show how to do variable elimination



Junction Trees to Variable Elimination:



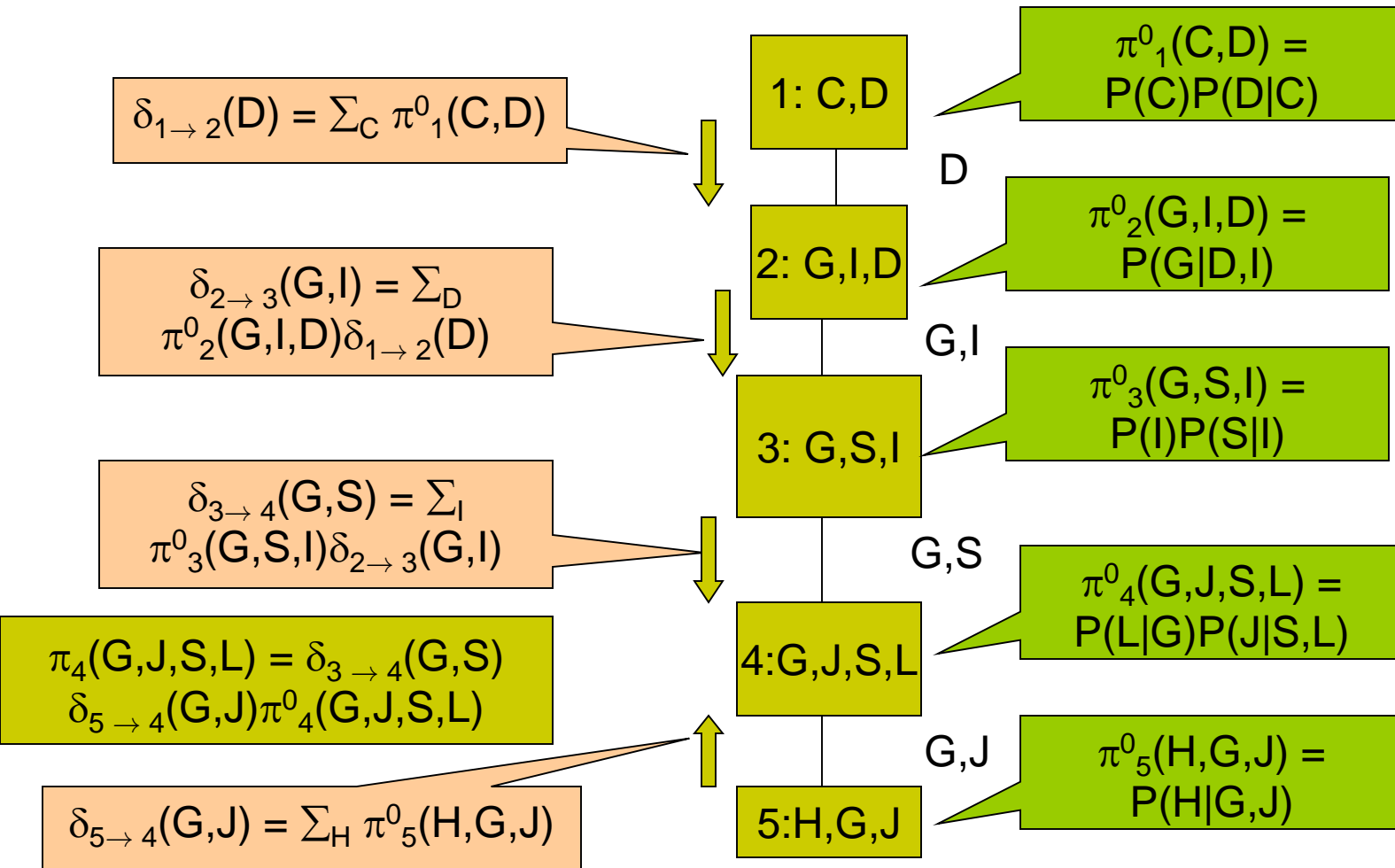
- Initialize potentials first:





Junction Trees to Variable Elimination:

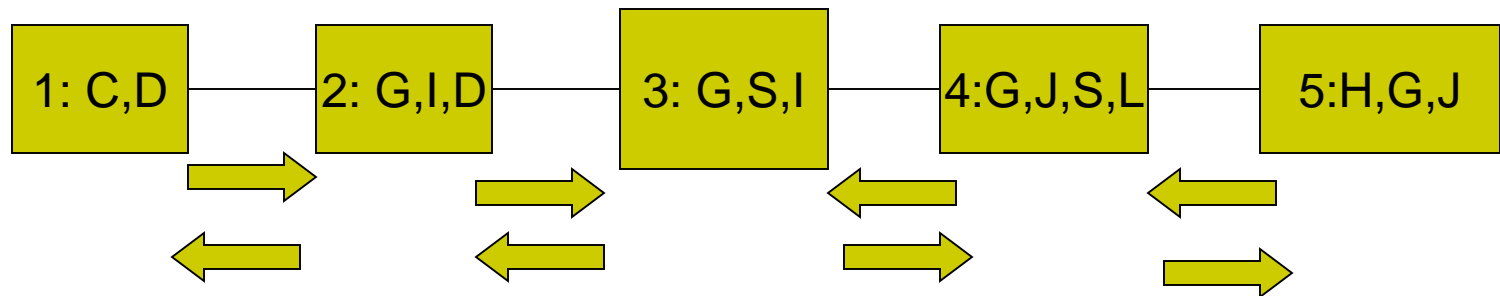
- Pass messages: (C_4 is the root)





Junction Tree calibration

- Aim is to compute marginals of each node using least computation
 - Similar to the 2-pass sum-product algorithm



- C_i transmits a message to its neighbor C_j after it receives messages from all other neighbors
- Called “**Shafer-Shenoy**” clique tree algorithm

Message passing with division

- Consider calibrated potential at node C_i whose neighbor is C_j

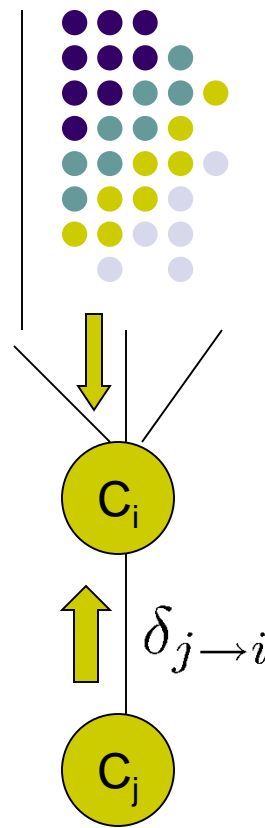
$$\pi_i = \pi_i^0 \prod_{k \in N(i)} \delta_{k \rightarrow i} = \pi_i^0 \times \delta_{j \rightarrow i} \times \prod_{k \in N(i) - j} \delta_{k \rightarrow i}$$

- Consider message from C_i to C_j

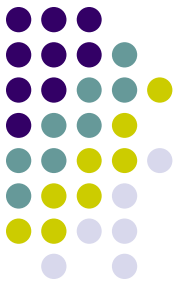
$$\delta_{i \rightarrow j} = \sum_{C_i - S_{ij}} \pi_i^0 \prod_{k \in N(i) - j} \delta_{k \rightarrow i}$$

- Hence, one can write:

$$\delta_{i \rightarrow j} = \sum_{C_i - S_{ij}} \frac{\pi_i}{\delta_{j \rightarrow i}}$$



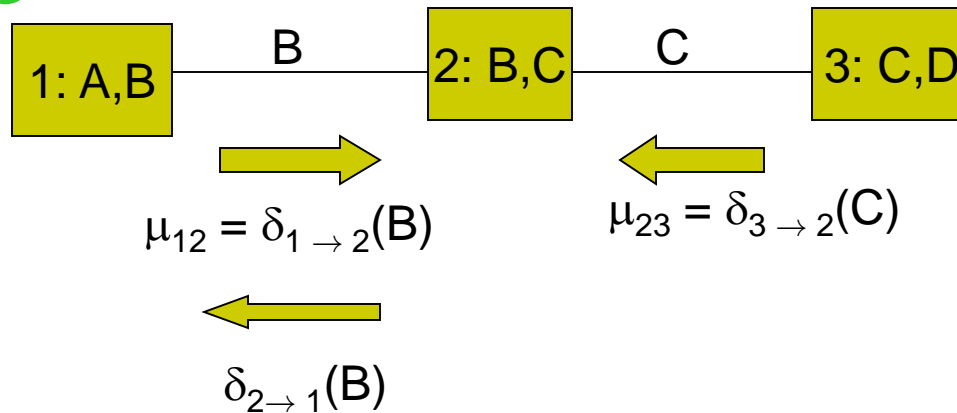
Message passing with division



- **Belief-update** or **Lauritzen-Spiegelhalter** algorithm
 - Each cluster C_i maintains its fully updated current beliefs π_i
 - Each sepset s_{ij} maintains μ_{ij} , the previous message passed between C_i - C_j regardless of direction
 - Any new message passed along C_i - C_j is divided by μ_{ij}

Belief Update message passing

Example



$$\pi_2(B, C) = \pi_2^0(B, C) \delta_{1 \rightarrow 2}(B) \delta_{3 \rightarrow 2}(C)$$

$$\delta_{2 \rightarrow 1}(B) = \sum_C \pi(B, C)$$

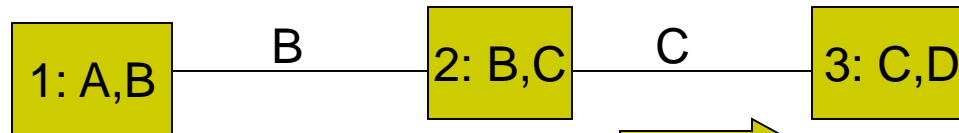
Actual message

$$m_{2 \rightarrow 1} = \frac{\delta_{2 \rightarrow 1}}{\mu_{12}} = \frac{\sum_C \pi(B, C)}{\mu_{12}} = \sum_C \frac{\pi_2^0(B, C) \delta_{1 \rightarrow 2}(B) \delta_{3 \rightarrow 2}(C)}{\delta_{1 \rightarrow 2}(B)}$$
$$= \sum_C \pi_2^0(B, C) \delta_{3 \rightarrow 2}(C)$$

This is what we expect to send in the regular message passing!

Belief Update message passing

Another Example



$$\delta_{2 \rightarrow 3}(C) = \mu_{23}^0$$

$$\delta_{3 \rightarrow 2}(C) = \mu_{23}^1$$

$$\delta_{2 \rightarrow 3}(C) = \sum_B \pi^0(B, C) = \mu_{23}^0$$

$$\pi_3(C, D) = \delta_{2 \rightarrow 3}(C) \pi_3^0(C, D)$$

$$\delta_{3 \rightarrow 2}(C) = \sum_D \pi_3(C, D) = \mu_{23}^1$$

$$\begin{aligned} m_{3 \rightarrow 2}(C) &= \frac{\delta_{3 \rightarrow 2}(C)}{\mu_{23}^0} = \frac{\sum_D \pi_3(C, D)}{\mu_{23}^0} \\ &= \frac{\sum_D \delta_{2 \rightarrow 3}(C) \pi_3^0(C, D)}{\delta_{2 \rightarrow 3}(C)} = \sum_D \pi_3^0(C, D) \end{aligned}$$

This is exactly the message C_2 would have received from C_3 if C_2 didn't send an uninformed message: **Order of messages doesn't matter!**

Belief Update message passing

Junction tree invariance



- Recall: Junction Tree measure:

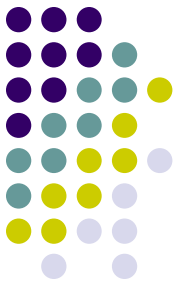
$$P(\mathbf{X}) = \frac{\prod_{\mathbf{C}_i \in T} \pi_i[\mathbf{C}_i]}{\prod_{(i,j) \in T} \mu_{ij}(\mathbf{S}_{ij})}$$

- A message from \mathbf{C}_i to \mathbf{C}_j changes only π_j and

$$\mu_{ij}: \quad \pi'_j = \pi_j \frac{\mu'_{ij}}{\mu_{ij}} \quad \Rightarrow \quad \frac{\pi_j}{\mu_{ij}} = \frac{\pi'_j}{\mu'_{ij}}$$

- Thus the measure remains unchanged for updated potentials too!

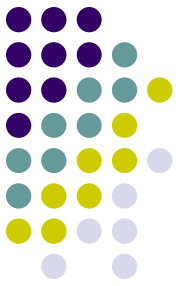
Junction trees from Chordal graphs



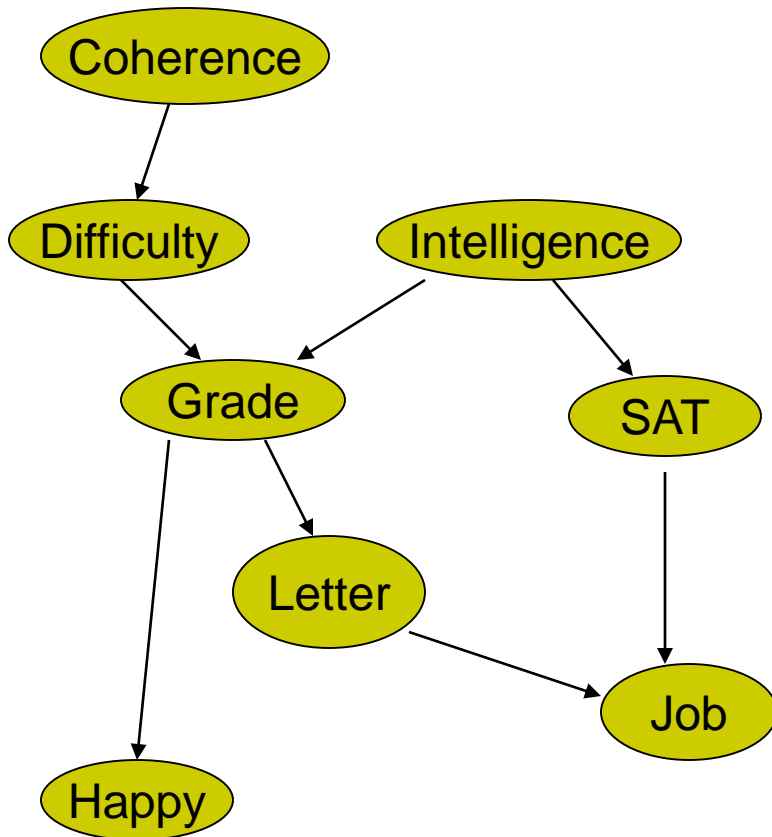
- Recall: A junction tree can be obtained by the induced graph from variable elimination
- Alternative approach: using **chordal graphs**
- Recall:
 - Any chordal graph has a clique tree
 - Can obtain chordal graphs through **triangulation**
- Finding a minimum triangulation, where largest clique has minimum size is NP-hard

Junction trees from Chordal graphs

Maximum spanning tree algorithm



- Original Graph

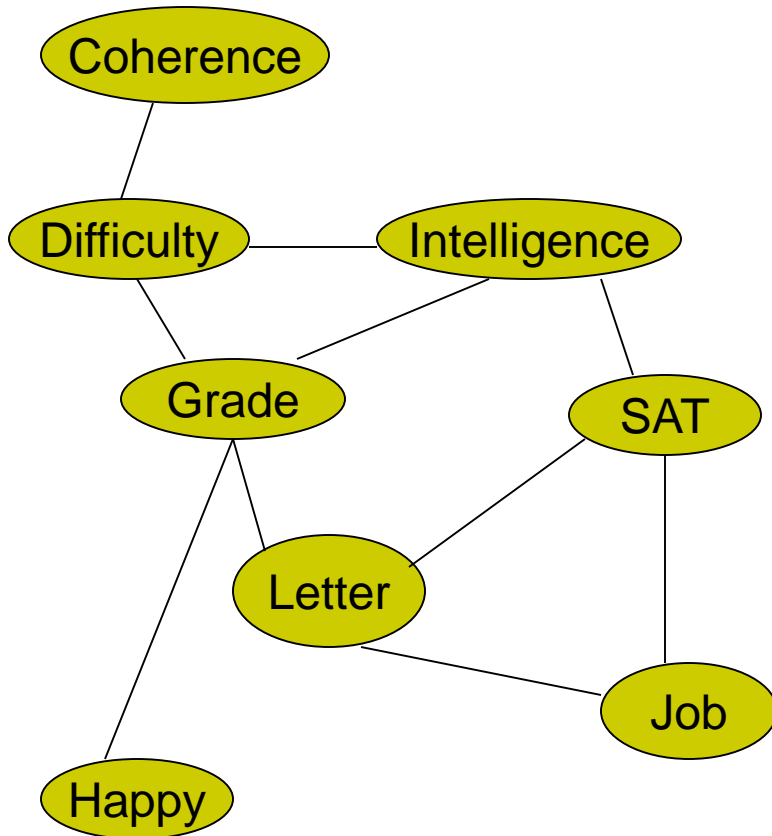


Junction trees from Chordal graphs

Maximum spanning tree algorithm



- Undirected moralized graph

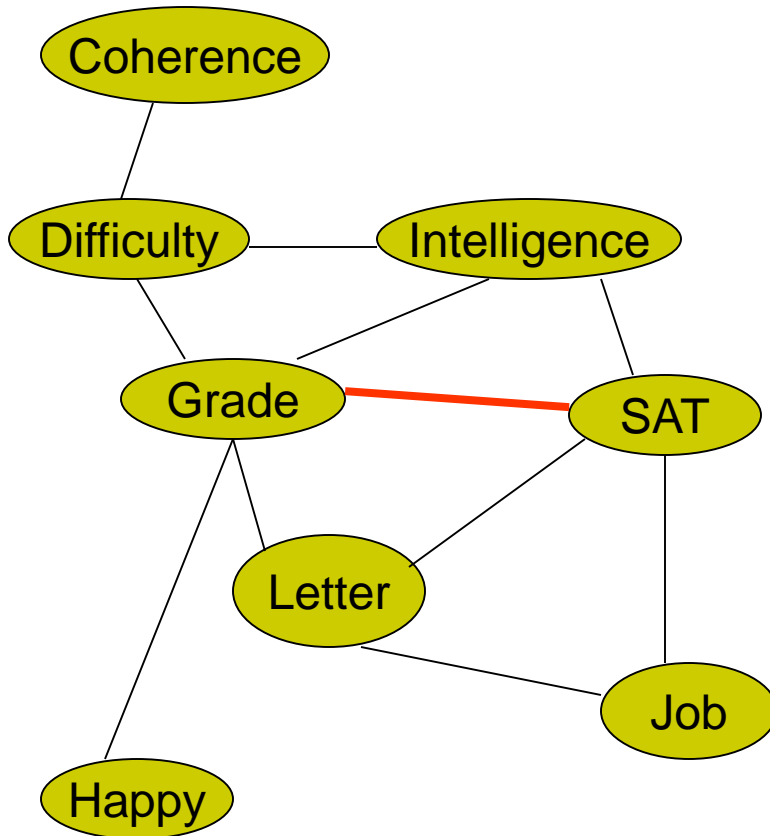


Junction trees from Chordal graphs

Maximum spanning tree algorithm

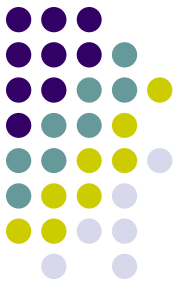


- Chordal (Triangulated) graph

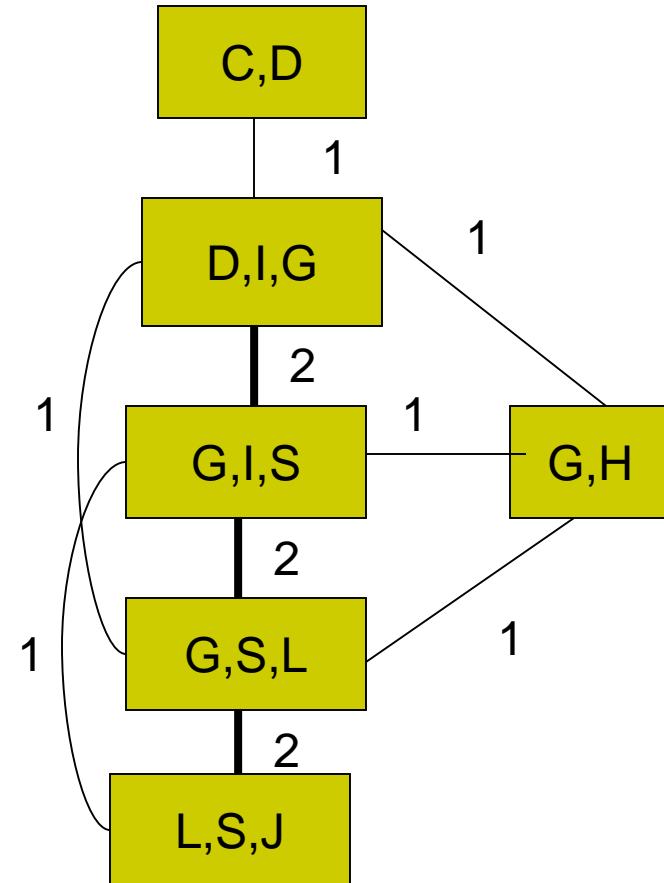
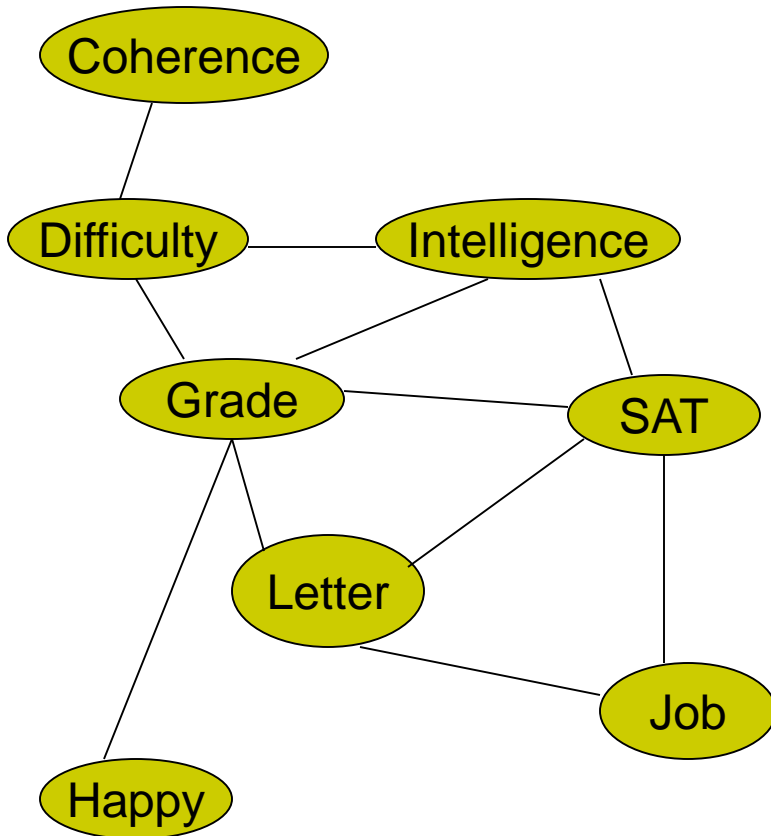


Junction trees from Chordal graphs

Maximum spanning tree algorithm



- Cluster graph

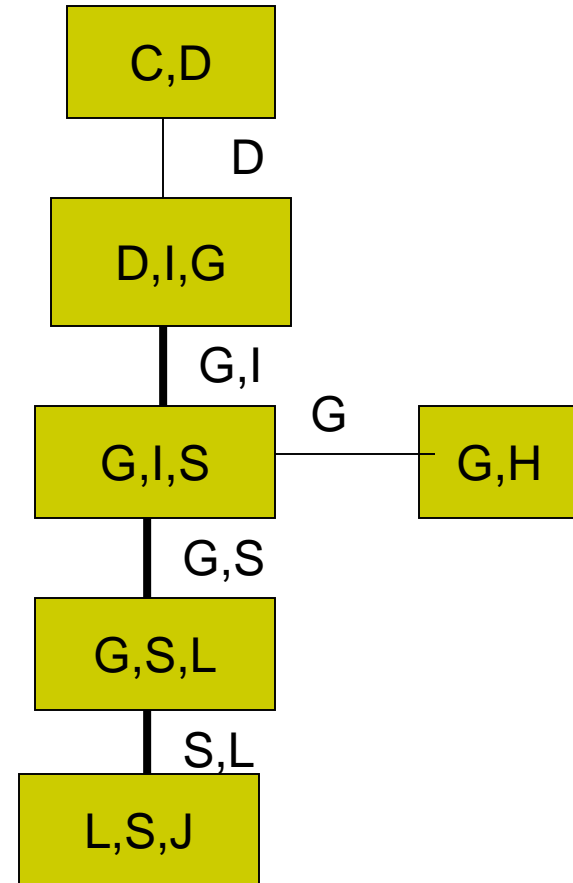
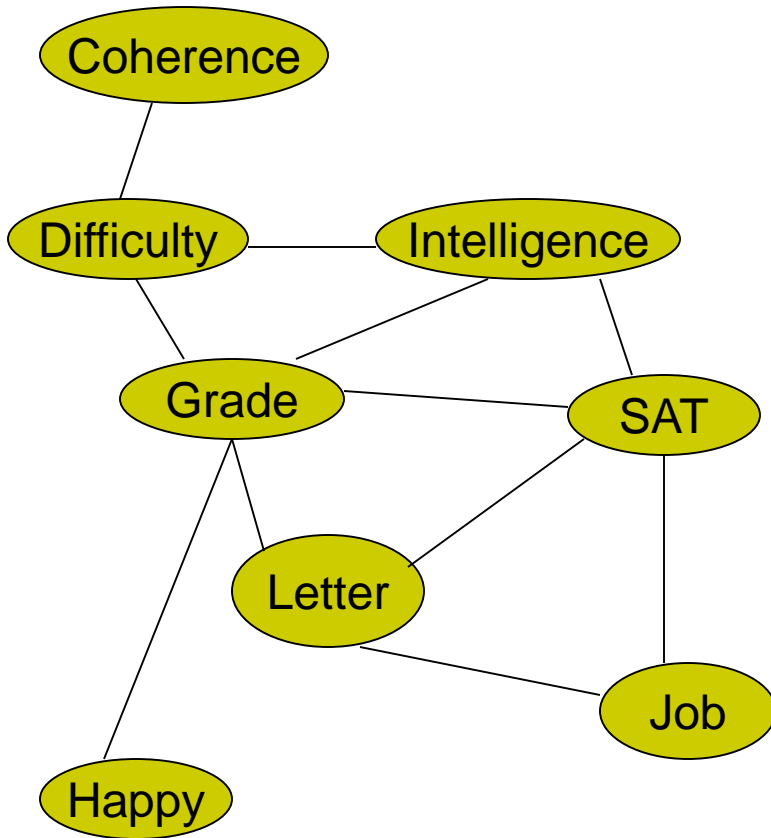


Junction trees from Chordal graphs

Maximum spanning tree algorithm



- Junction tree





Summary

- Junction tree data-structure for exact inference on general graphs
- Two methods
 - Shafer-Shenoy
 - Belief-update or Lauritzen-Speigelhalter
- Constructing Junction tree from chordal graphs
 - Maximum spanning tree approach